ON-LINE ALGORITHMS AND REVERSE MATHEMATICS

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Abstract

In this thesis, we classify the reverse-mathematical strength of sequential problems. If we are given a problem P of the form

$$\forall X(\alpha(X) \rightarrow \exists Z\beta(X, Z))$$

then the corresponding *sequential problem*, SeqP, asserts the existence of infinitely many solutions to P:

$$\forall X\,(\forall n\alpha(X_n) \rightarrow \exists Z\forall n\beta(X_n, Z_n))$$

P is typically provable in $\mathsf{RCA_0}$ if all objects involved are finite. SeqP, however, is only guaranteed to be provable in $\mathsf{ACA_0}$. In this thesis we exactly characterize which sequential problems are equivalent to $\mathsf{RCA_0}$, $\mathsf{WKL_0}$, or $\mathsf{ACA_0}$.

We say that a problem P is solvable by an *on-line algorithm* if P can be solved according to a two-player game, played by Alice and Bob, in which Bob has a winning strategy. Bob wins the game if Alice's sequence of plays $\langle a_0, \ldots, a_k\rangle$ and Bob's sequence of responses $\langle b_0, \ldots, b_k\rangle$ constitute a solution to P. Formally, an on-line algorithm A is a function that inputs an admissible sequence of plays $\langle a_0, b_0, \ldots, a_j\rangle$ and outputs a new play b_j for Bob. (This differs from the typical definition of "algorithm," though quite often a concrete set of instructions can be easily deduced from

A.)

We show that SeqP is provable in $\mathsf{RCA_0}$ precisely when P is solvable by an on-line algorithm. Schmerl [33] proved this result specifically for the graph coloring problem; we generalize Schmerl's result to any problem that is on-line solvable. To prove our separation, we introduce a principle called $\mathsf{Predict}_k(r)$ that is equivalent to $\neg\mathsf{WKL_0}$ for standard k, r.

We show that $\mathsf{WKL_0}$ is sufficient to prove SeqP precisely when P has a solvable closed kernel. This means that a solution exists, and each initial segment of this solution is a solution to the corresponding initial segment of the problem. (Certain bounding conditions are necessary as well.) If no such solution exists, then SeqP is equivalent to $\mathsf{ACA_0}$ over $\mathsf{RCA_0} + \mathsf{I}\Sigma^0_2$; $\mathsf{RCA_0}$ alone suffices if only sequences of standard length are considered. We use different techniques from Schmerl [34] to prove this separation, and in the process we improve some of Schmerl's results on Grundy colorings.

In Chapter 4 we analyze a variety of applications, classifying their sequential forms by reverse-mathematical strength. This builds upon similar work by Dorais [6] and Hirst and Mummert [22]. We consider combinatorial applications such as matching problems and Dilworth's theorems, and we also consider classic algorithms such as the task scheduling and paging problems. Tables summarizing our findings can be found at the end of Chapter 4.

Acknowledgments

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Contents

Chapter 1

Background

1.1 Reverse Mathematics

Reverse Mathematics is a program in mathematical logic that intends to determine which axioms are necessary to prove a given theorem. Most mathematical activity starts by assuming a set of axioms in order to deduce a theorem; we tend to go in "reverse," starting with a theorem and proving the necessary axioms.

Stephen Simpson, among the first significant contributors to the field, poses as the "Main Question" of Reverse Mathematics ([35], p. 2): "Which set existence axioms are needed to prove the theorems of ordinary, non-set-theoretic mathematics?" Clearly much of mathematics can be proven without the full strength of, say, the Axiom of Choice, where we quantify over every possible set. No, most of the best-known theorems require only real numbers, sets of real numbers, sets of sets of real numbers, and maybe an order or two higher at best.

So we examine set existence axioms that are much weaker than the full ZFC, but our program provides a rich refinement of "ordinary" mathematics according to strength and implication. Friedman [11] developed a hierarchy of axiom systems in his paper *"Some systems of second-order arithmetic and their use,"* and this hierarchy has been the core of the Reverse Mathematics program ever since. Nearly all of the reverse mathematics background can be found in Simpson's comprehensive monograph [35]. Hirschfeldt's recent textbook [19] is another valuable resource, particularly for the reverse mathematics of combinatorial principles.

Friedman's axiom systems are based in computability theory; the computability background can be found in the standard textbooks [30] and [38]. We use Φ_e to denote the e^{th} partial computable function $\Phi_e : \mathbb{N} \to \mathbb{N}$ in a fixed enumeration; Φ_e^A is the e^{th} partial computable function relative to the oracle A. We write $\Phi_e(n)\downarrow[s]$ to mean that the function halts at stage s; this formally means that s encodes a computation via Φ_e that inputs n and outputs a certain natural number $m = \Phi_e(n)$.

The halting set $\emptyset'$ is defined as $\emptyset' = \{e : \Phi_e(e)\downarrow\}$. Given a set A, the halting set relative to the oracle A, also called the Turing jump of A, is defined as $A' = \{e : \Phi_e^A(e)\downarrow\}$.

We will sometimes use the notation $f \subseteq: A \to B$ to denote that $\operatorname{dom} f \subseteq A$ and $\operatorname{ran} f \subseteq B$. If $r \in \mathbb{N}$, then the notation $f : X \to r$ means that $r = \{0, 1, \ldots, r-1\}$. D_e denotes "the e^{th} finite set"; formally, if $A = \{a_0, a_1, \ldots, a_k\}$, then $A = D_e$ where $e = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_k}$.

In most mathematical writing, the symbols $\mathbb{N}$ and ω are used interchangeably for the natural numbers. Since we will frequently work in nonstandard models, the numbers in our model may not behave like the standard natural numbers. Throughout this thesis, the symbol $\mathbb{N}$ will refer to the set of numbers in our model, and ω will refer to the set of standard natural numbers in our model.

Second-order arithmetic and the Big Five axiom subsystems

Reverse mathematics takes place in *second-order arithmetic*. The language of second-order arithmetic is two-sorted, including number variables $(x, y, z, \ldots)$ and set variables $(X, Y, Z, \ldots)$. Number variables are intended to range over $\mathbb{N}$ and set variables are intended to range over subsets of $\mathbb{N}$. The language also includes $+$, $\cdot$, 0, 1, $=$, $<$, $\in$, and the standard logical connectives, number quantifiers, and set quantifiers. It is easy to encode real numbers and functions $f : \mathbb{N} \to \mathbb{N}$ as sets of integers. We can further encode many familiar mathematical objects (Borel sets, continuous functions $g : \mathbb{R} \to \mathbb{R}$, separable metric spaces, graphs and other combinatorial objects) and can state many of their theorems in the language of second-order arithmetic.

The axioms of second-order arithmetic include Robinson arithmetic (which is Peano arithmetic minus full induction), comprehension axioms for each formula φ of the language, and induction axioms for each formula φ of the language, where φ may include free number and set variables as parameters. However, even this is usually more axioms than necessary, and so we often restrict ourselves to smaller subsystems.

Definition 1.1.1. The subsystem RCA_0 consists of Robinson arithmetic, comprehension axioms for all Δ^0_1-formulas, and induction axioms for all Σ^0_1-formulas.

RCA_0 (Recursive Comprehension Axiom) is a very mild set of assumptions, and thus RCA_0 is typically our *base theory*, the theory we assume when proving relationships among stronger statements.

Definition 1.1.2. A *tree* is a downward-closed subset $T \subseteq \mathbb{N}^{<\mathbb{N}}$. A *binary tree*, sometimes known as a *0–1 tree*, is a downward-closed subset $T \subseteq 2^{<\mathbb{N}}$.

Definition 1.1.3. The subsystem WKL_0 consists of RCA_0 plus Weak König's Lemma: Every infinite binary tree has an infinite path.

Definition 1.1.4. The subsystem ACA_0 consists of RCA_0 plus comprehension axioms for all arithmetical formulas, and induction axioms for all arithmetical formulas.

RCA_0, WKL_0, and ACA_0 are three of the "Big Five" subsystems. The two strongest, ATR and Π^0_1–CA_0, will not be used in this thesis, so I will not define them explicitly. We have the following strict implications:

$$\Pi^0_1\text{–}\mathsf{CA}_0 \Longrightarrow \mathsf{ATR}_0 \Longrightarrow \mathsf{ACA}_0 \Longrightarrow \mathsf{WKL}_0 \Longrightarrow \mathsf{RCA}_0$$

.

A substantial number of familiar theorems in analysis, algebra, and combinatorics are equivalent to one of the big five subsystems (over the base theory RCA_0). For just one example, the theorem "every countable vector space over $\mathbb{Q}$ has a basis" is equivalent to ACA_0 over RCA_0. The theorem can be deduced by ACA_0, but if you assume the theorem (and the base theory RCA_0), you can also *prove* ACA_0, showing that ACA_0 is necessary as well as sufficient. We could not have proved the theorem had we only assumed WKL_0.

There are some theorems whose equivalence to either WKL_0 or ACA_0 is very useful when trying to pinpoint another theorem's strength. Proofs of all parts of the two theorems below can be found in [35].

Theorem 1.1.5. *The following are equivalent over* RCA_0*:*

(i) WKL_0

(ii) Bounded König's Lemma: Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ *be an infinite tree. If there is a function* $g : \mathbb{N} \to \mathbb{N}$ *such that for all* $\sigma \in T$ *we have* $\sigma(m) < g(m)$*, then* T *has an infinite path.*

(iii) Σ^0_1*-separation: Let* $f, g : \mathbb{N} \to \mathbb{N}$ *be injective functions with disjoint ranges, i.e.,* $\forall m \, \forall n \, (f(m) \neq g(n))$*. Then there exists a set* X *that separates the ranges of* f *and* g*:*

$$\exists X \, \forall m \, (f(m) \in X \, \wedge \, g(m) \notin X)$$

Theorem 1.1.6. *The following are equivalent over* RCA_0*:*

(i) ACA_0

(ii) Σ^0_1*-comprehension*

(iii) The existence of the Turing jump A' *of any set* A

(iv) The existence of a range of an arbitrary function: Let $f : \mathbb{N} \to \mathbb{N}$ *be an injective function. Then there exists a set* $X \subseteq \mathbb{N}$ *such that*

$$\forall n (n \in X \, \leftrightarrow \, \exists m \, (f(m) = n))$$

Other important subsystems

Plenty of theorems do not neatly fall into the five-level hierarchy above. One theorem whose precise strength long eluded us was RT^2_2, the infinite Ramsey's Theorem for Pairs. We finally know, thanks to Jockusch [25] and a 2012 paper by Liu [29], that WKL_0 and RT^2_2 are incomparable, meaning that models exist of both $\mathsf{WKL}_0 + \neg\mathsf{RT}^2_2$ and of $\mathsf{RT}^2_2 + \neg\mathsf{WKL}_0$.

Other subsystems that will be important to our work are the induction and bounding schemes.

Definition 1.1.7. Let Γ be a class of formulas. The *induction scheme* for Γ, denoted

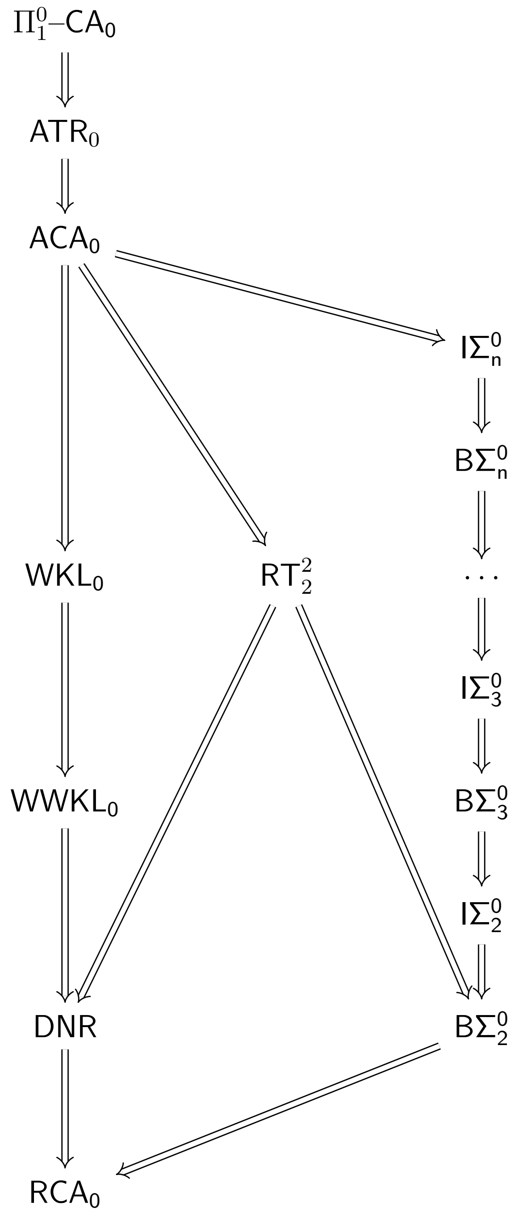

Figure 1.1: Implications of common subsystems. All arrows represent strict implications.

by $\mathsf{I}\Gamma$, is defined as the collection of all formulas

$$(\varphi(0) \wedge \forall n\,[\varphi(n) \rightarrow \varphi(n+1)]) \rightarrow \forall n\,\varphi(n)$$

for every formula $\varphi \in \Gamma$.

Definition 1.1.8. Let Γ be a class of formulas. The *bounding scheme* for Γ, denoted by $\mathsf{B}\Gamma$, is defined as the collection of all formulas

$$(\forall x < y)\,(\exists z)\,\varphi(x, z) \rightarrow (\exists b)\,(\forall x < y)\,(\exists z < b)\,\varphi(x, z)$$

for every formula $\varphi \in \Gamma$.

We will mostly be interested in the subsystems $\mathsf{I}\Sigma^0_n$ and $\mathsf{B}\Sigma^0_n$ for $n \geq 2$. The first three statements in the theorem below are proven in [17]; the fourth is due to Slaman [36].

Theorem 1.1.9 (RCA_0)**.** *Let $n \geq 1$.*

- $\mathsf{I}\Sigma^0_{n+1}$ *is strictly stronger than* $\mathsf{B}\Sigma^0_{n+1}$*, which is strictly stronger than* $\mathsf{I}\Sigma^0_n$*.*
- $\mathsf{I}\Sigma^0_n \Leftrightarrow \mathsf{I}\Pi^0_n$.
- $\mathsf{B}\Sigma^0_{n+1} \Leftrightarrow \mathsf{B}\Pi^0_n$.
- *If $n \geq 2$, then* $\mathsf{B}\Sigma^0_n \Leftrightarrow \mathsf{I}\Delta^0_n$*.*

$\mathsf{I}\Sigma^0_1$ is included in RCA_0, while every $\mathsf{I}\Sigma^0_n$ is included in ACA_0.

An equivalent formulation of $\mathsf{I}\Sigma^0_n$ that will be useful throughout this thesis is the least number principle for Π^0_n-formulas.

Proposition 1.1.10. $\mathsf{I}\Sigma^0_n$ *is equivalent to the least number principle for* Π^0_n*-formulas: For each* Π^0_n*-formula* ψ*, if* $\psi(n)$ *holds for some* n*, then there is a least* n *for which* $\psi(n)$ *holds.*

A final principle that will play a major role in this thesis is DNR, the existence of a diagonally recursive function.

Definition 1.1.11. A *diagonally non-recursive function* is a function $g : \mathbb{N} \to \mathbb{N}$ such that $\forall e(g(e) \neq \Phi_e(e))$.

So diagonally non-recursive functions avoid every partial recursive function at at least one point.

Definition 1.1.12. The principle DNR states that for each oracle A, there exists a function $g : \mathbb{N} \to \mathbb{N}$ which is diagonally non-recursive relative to A; i.e.,

$$\forall e \left(g(e) \neq \Phi^A_e(e)\right)$$

.

DNR is strictly stronger than $\mathsf{RCA_0}$, since the minimal model of $\mathsf{RCA_0}$, which consists of all Δ^0_1-sets, does not satisfy DNR. However, DNR is one of the weakest principles commonly studied in reverse mathematics:

Theorem 1.1.13 (Ambos-Spies [4])**.** DNR *is strictly weaker than* $\mathsf{WKL_0}$*. In fact, it is strictly weaker than Weak Weak König's Lemma (*$\mathsf{WWKL_0}$*).*

We note that the statement "$\Phi^A_e(n) = x$" is expressible in second-order arithmetic (see [19] p. 50) as follows: There exists s and $i, j, x < s$ such that $D_i \subseteq A$, $D_j \subseteq \mathbb{N} \setminus A$, and such that $\Phi_e\langle n, i, j, x\rangle \downarrow [s]$ (so that s codes a computation with information from e, n, D_i, D_j and gives output x). Similarly, we can define in second-order arithmetic that $B \leq_T A$ (B is Turing reducible to A) if there is e such that $n \in B$ if $\Phi_e^A(n) = 1$ and $n \notin B$ if $\Phi_e^A(n) = 0$. This allows us to express the Turing jump of A in second-order arithmetic as well: $e \in A'$ iff $\Phi_e^A(e)\downarrow$.

1.2 On-Line Algorithms

On-line algorithms are necessary when we are forced to ask the question, "what is the best decision I can make with zero knowledge of the future?" In any setting where we receive a sequence of requests, and must make an *immediate decision*, with no knowledge of the future requests, often even not knowing the *number* of future requests, we must turn to our optimal on-line algorithm.

(Note: the terminology "on-line" does not refer to the Internet. In fact, the term predates the wide availability of the Internet. There is no consensus on hyphenation; both "on-line" and "online" are used in the literature. Our major references in the reverse mathematics world do use the hyphen, so we will use it as well.)

It is most useful to view an on-line algorithm in terms of a two-player game. Throughout this thesis, the two players will be named Alice and Bob. Alice and Bob alternate plays:

$$\begin{array}{r|cccccccc} \text{Alice} & a_0 & & a_1 & & a_2 & & \cdots \\ \hline \text{Bob} & & b_0 & & b_1 & & b_2 & \cdots \end{array}$$

Bob is aspiring to construct an object satisfying a given relation with Alice's request sequence $\bar{a}$, and is frequently trying to minimize a certain cost parameter $C(\bar{a}, \bar{b})$ with his sequence $\bar{b}$ of plays. Alice, the adversary, is trying to choose requests that will defeat Bob — either preventing him from constructing his object at all, or attempting to maximize the cost $C(\bar{a}, \bar{b})$.

Games, strategies, and algorithms

If A, B are trees (see Definition 1.1.2), a *two-player game* G is a tree of sequences $\langle a_0,\, b_0,\, a_1,\, b_1,\, a_2,\, b_2,\, \ldots,\, \rangle$ with $\langle a_0, a_1, \ldots, a_i\rangle \in A$ and $\langle b_0, b_1, \ldots, b_i\rangle \in B$ for all i. The maximal paths in G (finite sequences or infinite branches) are sometimes called *outcomes* of the game. In this thesis we will call the two players Alice and Bob, and we will say that each a_j is a *play* for Alice and that each b_j is a *play* for Bob.

If A and B are trees then $A \otimes B$ denotes the set of all pairs $(\bar{a}, \bar{b})$ such that $lh(\bar{a}) = lh(\bar{b})$. Throughout this thesis, we will be considering relations $R \subseteq A \otimes B$.

The game $\mathsf{G}(A, B, R)$ will be defined as follows: $\mathsf{G}(A, B, R)$ consists of all $\bar{z} = \langle a_0, b_0, a_1, b_1, \ldots, a_k, b_k\rangle$ such that $\bar{a}\ R\ \bar{b}$ holds for every *proper* initial segment of $\bar{z}$. If $\bar{w} = \langle a_0, b_0, a_1, b_1, \ldots, a_k, b_k\rangle$ is a finite outcome of $\mathsf{G}(A, B, R)$, then the final relation $\bar{a}\ R\ \bar{b}$ fails. In this case, we say that outcome $\bar{w}$ constitutes a win for Alice. If $\bar{w} = \langle a_0, b_0, a_1, b_1, \ldots, \rangle$ is an infinite outcome of $\mathsf{G}(A, B, R)$, so that $\forall k\, \langle a_0, \ldots, a_k\rangle\ R\ \langle b_0, \ldots, b_k\rangle$, then we say that outcome $\bar{w}$ constitutes a *win* for player Bob.

We sometimes also say that Alice loses if $\bar{a} \notin A$; while our formal definitions do not allow this, this should not cause any confusion later.

Formally, an *on-line algorithm* is the same thing as a *strategy* for Bob in $\mathsf{G}(A, B, R)$: a function τ that will input any $\langle a_0,\, a_1,\, \ldots,\, a_j\rangle \in A$ and will output $b_j = \tau\langle a_0, a_1, \ldots, a_j\rangle$ such that

$$\Big\langle \tau\langle a_0\rangle,\, \tau\langle a_0, a_1\rangle,\, \ldots,\, \tau\langle a_0, a_1, \ldots, a_j\rangle \Big\rangle \in B.$$

(This differs from the most common definition of "algorithm" as a finite set of instructions, though in quite a few applications, we can easily obtain an on-line algorithm from an appropriate set of instructions for Bob.)

A *winning strategy* for Bob is a strategy τ such that for any j and for any $a_0, \ldots, a_j$, we have

$$\langle a_0, \ldots, a_j \rangle \; R \; \big\langle \tau\langle a_0 \rangle,\, \tau\langle a_0, a_1 \rangle,\, \ldots,\, \tau\langle a_0, a_1, \ldots, a_j \rangle \big\rangle.$$

A strategy for Alice is defined analogously; it is a function that inputs a sequence $\langle b_0, \ldots, b_{j-1} \rangle$ (or the empty sequence) and outputs a play $a_j = \sigma\langle b_0, \ldots, b_{j-1} \rangle$ such that

$$\big\langle \sigma\langle\rangle,\, \sigma\langle b_0 \rangle,\, \ldots,\, \sigma\langle b_0, \ldots, b_{j-1} \rangle \, \big\rangle \in A$$

A winning strategy for Alice is a strategy σ such that, for any infinite sequence $\langle b_0, b_1, b_2, \ldots, \rangle$ there exists a $j \in \mathbb{N}$ such that for every b_j with $\langle b_0, \ldots, b_j \rangle \in B$,

$$\big\langle \sigma\langle\rangle,\, \sigma\langle b_0 \rangle, \ldots,\, \sigma\langle b_0,\, \ldots, b_{j-1} \rangle \, \big\rangle \; R \; \langle b_0, \ldots, b_{j-1}, b_j \rangle \qquad \text{fails.}$$

If we assume all of second-order arithmetic, these games are determined, meaning that if Bob does not have a winning strategy, then Alice does. Indeed, if Bob does not have a winning strategy, then Alice can play a_0 such that Bob does not have a winning strategy above a_0, and in general there exists a_i that Alice can play such that Bob does not have a winning strategy above a_i. Thus this sequence $\bar{a}$ of plays will be a winning play for Alice.

Typically Bob does not get a "second chance" — if Alice defeats Bob in even one round, we declare the game over.

The opposite of an on-line algorithm is an *off-line algorithm*, in which Alice's future requests *are* known in advance. Formally this means that the algorithm will output b_j as a function of *all* values of $\{a_k : k \in \mathbb{N}\}$, not just as a function of a_0 through a_j. Usually we may refer to an optimal off-line algorithm OPT, which guarantees the minimum possible cost. OPT may or may not be unique in this way. It goes without saying that the optimal on-line and off-line algorithms typically have unequal cost.

Definition 1.2.1. Let A be an on-line algorithm. The *competitive ratio* of A, $\mathrm{CR}(A)$, is the highest possible ratio between the costs $C(\bar{a}, A(\bar{a}))$ and $C(\bar{a}, \mathrm{OPT}(\bar{a}))$ over all possible request sequences $\bar{a}$:

$$\mathrm{CR}(A) = \sup_{\bar{a}} \frac{C(\bar{a}, A(\bar{a}))}{C(\bar{a}, \mathrm{OPT}(\bar{a}))}.$$

Definition 1.2.2. Let A be an on-line algorithm. We say that A is d-competitive if there exists $K \in \mathbb{N}$ such that

$$\forall \bar{a}\ [\, C(\bar{a}, A(\bar{a})) \leq d \cdot C(\bar{a}, \mathrm{OPT}(\bar{a})) + K \,] .$$

On-line algorithms have a wide range of applications, several of which (scheduling, paging) will be presented in Chapter 4, but which also include computing resource management (e.g. the k-server problem, moving tasks between servers as new tasks are called), and computational finance (e.g., buying and selling stocks immediately as their valuation changes). In more sophisticated models, randomness can be introduced, and certainly there are settings where minimizing the *expected* cost is the way to go, rather than minimizing the (sometimes quite unlikely) worst case scenario. For further reading, see the survey articles [2], [3], and [27].

Example: On-line graph colorings

Definition 1.2.3. A *graph* is a pair (V, E) where $V \subseteq \mathbb{N}$ is a set of vertices and $E \subseteq V \times V$ is an irreflexive, symmetric relation.

Let $G = (V, E)$ be a graph. A *proper r-coloring* of G is a function $\varphi : V \to r$ such that $\varphi(x) \neq \varphi(y)$ whenever $(x, y) \in E$.

A *universal class* of graphs $\mathcal{C}$ is a class that is closed under isomorphisms, such that for every graph G, $G \in \mathcal{C}$ if and only if every finite induced subgraph of G is in $\mathcal{C}$. This includes all classes of the form $\mathrm{Forb}(\mathcal{F})$; the class of graphs that have no induced subgraphs in a particular class of finite graphs $\mathcal{F}$. (So that the graphs in $\mathcal{F}$ are "forbidden.") For example, the universal class of graphs with no 3-cycles is equal to $\mathrm{Forb}(K_3)$, where K_n is the complete graph on n vertices. $\mathrm{Forb}(K_3)$ is sometimes called the class of triangle-free graphs.

Let $\mathcal{C}$ be a universal class of graphs, and let $r \geq 2$. The game associated with on-line r-coloring of graphs in $\mathcal{C}$, which we call $\mathsf{G}(\mathcal{C}, r)$, is played out as follows:

- Alice plays a new graph vertex and specifies whether or not it is connected by an edge with each vertex that she played on an earlier round. Alice loses immediately if the graph played thus far does not belong to the class $\mathcal{C}$.

- Bob responds by assigning a color from $\{0, \ldots, r-1\}$ to the vertex that Alice just played. Bob loses immediately if the colors assigned thus far do not form a proper r-coloring of the graph.

Bob wins if the game goes on indefinitely without either player losing. This definition of "game" can be made to conform to our earlier formal definition if Alice plays a number a_0 that codes the vertex and additional edges that she adds at each stage.

Definition 1.2.4. If Bob has a winning strategy for this game $\mathsf{G}(\mathcal{C}, r)$, we say that the class $\mathcal{C}$ is *on-line r-colorable.*

In the context of graph theory, a *tree* will always mean a cycle-free graph. A *forest* is a graph whose components are all trees. It is trivial that every forest is 2-colorable — just take the levels in each tree component and color the odd levels red and the even levels blue. However, the class of forests is NOT on-line colorable.

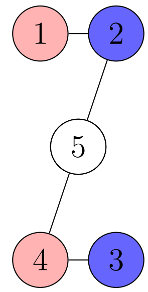

Figure 1.2: A forest is not on-line 2-colorable.

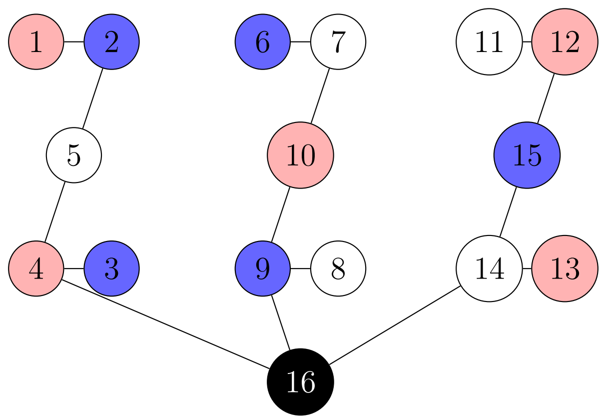

Figure 1.3: A forest is not on-line 3-colorable.

Proposition 1.2.5. *The class of forests is not on-line r-colorable for any $r \geq 2$.*

Figure 1.2 and Figure 1.3 show the counterexamples for 2-colorings and 3-colorings. Here Alice plays each vertex in numerical order, and Bob responds with a color (shaded in the figure). When Alice plays vertex 5, she forces Bob to play a third color by playing a vertex with a red neighbor and a blue neighbor. Note that if F_2, F_3 are the forests pictured, then for each k, $F_2 \cap \{1, \ldots, k\}$ and $F_3 \cap \{1, \ldots, k\}$ are forests. It should be clear from the two figures how to inductively define a counterexample for $r > 3$ colors.

Let P_n be a path of length n, meaning that P_n has exactly n vertices $v_1, \ldots, v_n$ with $(v_i, v_{i+1}) \in E$ for $1 \leq i < n$, but $(v_i, v_j) \notin E$ for $i + 1 < j$.

Proposition 1.2.6 (Gyárfás and Lehel [16]). *Define $\mathcal{C}$ to be the class of all bipartite* *graphs in* $\text{Forb}(P_6)$*. For any number of colors* $r \geq 2$*,* $\mathcal{C}$ *is* r*-colorable but is not on-line* r*-colorable.*

Proof. All bipartite graphs are 2-colorable. For the on-line case, we construct a series of graphs G_k; see Figure 1.4 for G_2 and G_3. G_1 is a point; G_2 connects 2 copies of G_1. G_3 connects a new vertex x_3 (at the bottom) to two copies of G_1 and two copies of G_2, but in the first copy of G_2, x_3 connects to all vertices in the first bipartition, and in the other it connects to all vertices in the second bipartition. G_4 and the general G_n are constructed in the same way, connecting a new vertex x_n to two copies each of $G_1, G_2, \ldots, G_{n-1}$, connecting x_n to one of the bipartitions in one copy and to the other bipartition in the other copy.

It is easy to show inductively that G_n does not contain a copy of P_6 (notice that G_3 does have a subgraph isomorphic to P_5). We argue by strong induction that Alice can force Bob to play n different colors in G_n. Assume this is true of $G_1, \ldots, G_{n-1}$. Then when creating G_n, Alice will create two copies of G_1, followed by two copies of G_2, and so on, until she creates two copies of G_{n-1}. When Bob chooses to play the k^{th} distinct color on vertex v, Alice will force v to be in the first or second copy of G_k depending on what bipartition v is in. Then when Alice plays vertex x_n at the bottom, Bob is forced to play color n.

□

Example: On-line task scheduling

We give a short overview of another problem that we will explore in more detail in Section 4.4. In this problem, we schedule a series of simultaneous tasks on k multiprocessors. Tasks of varying processing time are announced by Alice, and Bob must immediately choose one of his k machines for each task. Bob's goal is to minimize

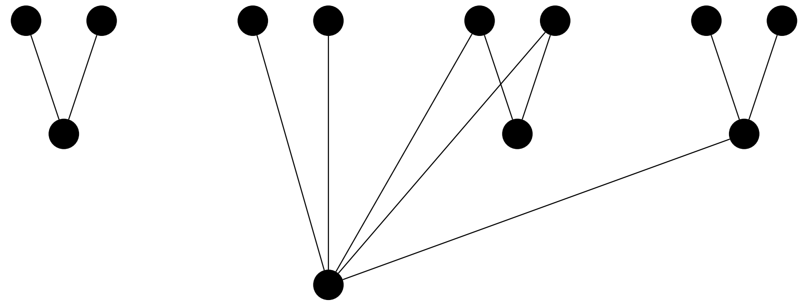

Figure 1.4: The graphs G_2 and G_3 in the proof of Proposition 1.2.6.

3		
1		
1	3	
1	3	6

6		
3	3	3
1	1	1

Figure 1.5: The optimal off-line solution (left) and on-line solution (right)

the total *completion time*, also called the *makespan*, which is the total time required to complete all tasks. The makespan plays the role of "cost parameter" C described above, with Bob seeking to minimize it and Alice seeking to maximize it. (This is contrary to the graph coloring problem, where Alice seeks to prevent a proper coloring entirely.) Of course, Bob has no knowledge the of future tasks' processing times, or (usually) how many more there will be, and he must optimize his choices accordingly.

Let us assume that there are $k = 3$ processors, and the sequence of tasks have processing times $(1, 1, 1, 3, 3, 3, 6)$. In Figure 1.5, each column represents a processor, and the numbers are the processing times of tasks assigned to that processor. The optimal processing time is clearly 6 — just schedule according to the left side of Figure 1.5. However, if the process stops before the final task, the optimal assignment (for the entire process) is no longer optimal for the truncated process. Even after two tasks, the optimal algorithm for the entire process places two tasks (of time 1) on the left processor and zero on the others, which is *double* the optimal makespan for the truncated process. That is, if we declare that Alice wins the game if the competitive ratio is ≥ 2, then Bob cannot schedule these tasks optimally at every stage of the game, for Alice would win after two rounds.

Nevertheless, Bob can win against this sequence of plays by Alice, maintaining a competitive ratio less than 2 at every stage. Any optimal on-line algorithm must schedule the first three tasks (the 1's) on three different processors. It is not hard to see that Bob's best on-line strategy is on the right side of Figure 1.5. The competitive ratio is thus $10/6 \approx 1.667$.

This example alone shows that there is no hope of finding an on-line task scheduling algorithm with competitive ratio $\leq 5/3$. The most naive on-line algorithm is to schedule each task on the machine with the current lightest load. Graham [14] showed that this algorithm has competitve ratio $2 - \frac{1}{k}$ where k is the number of processors. Combining these results shows that there is an on-line algorithm with competitive ratio in (5/3, 2]; as we will see in Section 4.4, Albers has improved these bounds to $(1.852, 1.923]$.

Chapter 2

Introduction to Sequential Forms and Prediction Principles

In Section 2.1, we introduce the main object we will be classifying, the sequential form of a problem. We give some motivation and history, including the connection with intuitionistic subsystems, where the strength of the sequential problem relates to the question of whether we must use the Law of Excluded Middle to prove that the problem is solvable.

In Section 2.2 we present the proofs of two examples: the sequential pigeonhole principle and the sequential bipartition problem. Technically this section is redundant, since the full classification theorem is proven in Chapter 3, but it illustrates the method in the context of a specific sequential problem rather than in full generality.

Section 2.3 introduces our two prediction principles $\mathsf{Predict}_k(r)$ and $\mathsf{Evade}_k(r)$. They are variants of $\mathsf{DNR}(k)$, and will be valuable tools in proving that certain sequential forms require $\mathsf{WKL_0}$. They were introduced (not by name) in Schmerl's paper [33] on the reverse mathematics of on-line graph colorings, where he showed that by assuming $\neg\mathsf{WKL_0}$, we can find an infinite forest that is not 2-colorable. We discuss this example in Section 2.4.

2.1 Introduction to Sequential Forms

Many theorems in combinatorics, and in mathematics generally, are of the form

$$\forall X(\alpha(X) \to \exists Z\beta(X, Z)) \tag{†}$$

Consider the statement, "every finite graph without an odd cycle is bipartite": it has the above form with $\alpha(X)$ stating "X is a finite graph with no odd cycles," and $\beta(X, Z)$ stating "Z is a bipartition of X." Finite versions of Ramsey's Theorem, the pigeonhole principle, and the existence of a graph coloring are also statements of this form.

Definition 2.1.1. Consider a statement of the form

$$\forall X(\alpha(X) \ \to \ \exists Z\beta(X, Z))$$

with X, Z both set variables. The *sequential form* of this statement is the following:

$$\forall X\,(\forall n\alpha(X_n) \ \to \ \exists Z\forall n\beta(X_n, Z_n))$$

where $X = \langle X_n\rangle_{n\in\mathbb{N}}$ and $Z = \langle Z_n\rangle_{n\in\mathbb{N}}$.

According to the sequential form, if we are given a sequence $X = \langle X_n\rangle_{n\in\mathbb{N}}$ of objects in the theorem's hypothesis (e.g. bipartite graphs) then there exists a sequence $Z = \langle Z_n\rangle_{n\in\mathbb{N}}$ of witnesses for the theorem's conclusion (e.g. bipartitions).

If the objects in question are finite, then the nonsequential statement is almost certainly provable in RCA_0, since one can computably find the finite object Z and verify the desired property. However, the sequential form, where we effectively prove the finite theorem infinitely often, is often stronger, though ACA_0 is always sufficient. We show this in Proposition 2.1.4; the proposition's hypothesis, that ACA_0 suffices to prove the nonsequential statement, is true of every statement we will consider. Also, recall from section 1.1 that D_n is the n^{th} finite set in a canonical ordering; specifically, if $A = \{a_0, \ldots, a_k\}$, then $A = D_n$ where $n = 2^{a_0} + \cdots + 2^{a_k}$.

Definition 2.1.2 (RCA_0)**.** A *finite set* is a set X such that $\exists k \forall i (i \in X \rightarrow i < k)$.

Definition 2.1.3 (RCA_0)**.** A *sequence of finite sets* $\langle X_n \rangle_{n \in \mathbb{N}}$ is a subset $A \subseteq \mathbb{N} \times \mathbb{N}$ such that for each $n \in \mathbb{N}$, $X_n = \{x : (n, x) \in A\}$ is finite.

Proposition 2.1.4. *Let $\varphi(A, B)$ be an arithmetical formula. Suppose that $\mathsf{ACA}_0 \vdash$ if A is a finite set, there exists a finite set B such that $\varphi(A, B)$. Then $\mathsf{ACA}_0 \vdash$ if $\langle X_n \rangle_{n \in \mathbb{N}}$ is a sequence of finite sets, there exists a sequence of finite sets $\langle Y_n \rangle_{n \in \mathbb{N}}$ such that $\forall n \, \varphi(A_n, B_n)$.*

Proof. Assume the hypothesis. We can use ACA_0 to define a function $b : \mathbb{N} \rightarrow \mathbb{N}$ that gives an upper bound $b(n)$ for each finite set X_n:

$$b(n) = m \leftrightarrow \big[(m = 0 \wedge \forall x (x \notin X_n)) \vee (m - 1 \in X_n \wedge (\forall k \geq m)(k \notin X_n)) \big]$$

Then, using the upper bounds, we can find a primitive recursive function f that codes each finite set X_n as $D_{f(n)}$. ($f(n)$ can be obtained from one of the finite subsets of $\{0, \ldots, b(n)\}$). Then we can use minimization to find the least e_n such that $\varphi(D_{f(n)}, D_{e_n})$ holds. Define $Y_n := D_{e_n}$ and we are done.

$\square$

Finding upper bounds for each X_n is crucial, since it allows us to answer questions of the form $(\forall x \in X_n) \, \psi(x)$ or $(\exists x \in X_n) \, \psi(x)$. In fact, the problem of finding a sequence of upper bounds is in itself equivalent to ACA_0:

Proposition 2.1.5. *The following are equivalent over* RCA_0*:*

(i) ACA_0

(ii) Given a sequence $\langle X_n \rangle_{n\in\mathbb{N}}$ *of finite sets, there is a sequence of upper bounds* $\langle b_n \rangle_{n\in\mathbb{N}}$ *such that* $\forall n \, \forall x \, (x \in X_n \rightarrow x \leq b_n)$.

Proof. (i) implies (ii) by the proof of Proposition 2.1.4. Assume (ii), and let A be an oracle. We will show that the halting set relative to A exists, implying ACA_0. Define $\langle X_n \rangle_{n\in\mathbb{N}}$ by:

$$\begin{array}{lcll} X_n & = & \{s\} & \text{if} \quad \Phi_n^A(n)\!\downarrow\![s] \ \wedge \ (\forall s' < s) \, \neg(\Phi_n^A(n)\!\downarrow\![s']) \\ X_n & = & \varnothing & \text{otherwise} \end{array}$$

Note that $\langle X_n \rangle_{n\in\mathbb{N}}$ is a computable sequence of sets. Then let $\langle b_n \rangle_{n\in\mathbb{N}}$ be the sequence of bounds guaranteed by (ii). We can define a function $h : \mathbb{N} \rightarrow \{0,1\}$ by: $h(n) = 1$ if there is $s \leq b_n$ with $\Phi_n^A(n) \downarrow [s]$, $h(n) = 0$ otherwise. Then $h(n) = 1$ if and only if $\Phi_n^A(n) \downarrow$; therefore, we can compute the Turing jump A', implying ACA_0. $\square$

The above proof makes use of the halting stage s of a Σ_1^0-function to create a witnessing example. This technique will play an important role in this thesis. It can be counterintuitive, since $\langle X_n \rangle_{n\in\mathbb{N}}$ is computable in that we can computably decide whether a given $k \in X_n$, but no computable function can decide whether each X_n has cardinality 0 or 1.

2.1.1 Relationship with intuitionistic subsystems

Sequential forms help formalize the notion of a constructive proof. Intuitively, if our nonsequential statement of form (†) can be proved constructively, then the sequential form is provable in $\mathsf{RCA_0}$, for there exists a uniform algorithm that will find the full sequence $\langle Z_n \rangle_{n\in\mathbb{N}}$ of witnesses for any infinite sequence $\langle X_n \rangle_{n\in\mathbb{N}}$ of inputs. Conversely, if the sequential form is strictly stronger than $\mathsf{RCA_0}$, that points to a certain lack of uniformity in the original proof.

Making this notion precise involves considering subsystems with and without LEM, the law of excluded middle. Subsystems that include LEM are called *classical*, and subsystems that do not include LEM are called *intuitionistic*; all axiom subsystems we have discussed have been classical.

Definition 2.1.6. LEM is the Law of Excluded Middle, which is the schema:

$$\forall x(\varphi(x) \vee \neg\varphi(x))$$

with an instance for each formula $\varphi(x)$.

Intuitively, if a statement in the form (†) can be proved without LEM, then there exists a constructive proof of this statement. Without the law of excluded middle, we cannot break our argument up into "cases" and prove each instance separately according to those cases; there must be a uniform procedure that will take any possible input x and produce a correct witness z.

The analysis of proofs with and without LEM requires techniques from proof theory, specifically the modified realizability of functions introduced by Kleene [28]. This is outside the scope of this thesis, but see Definition 2.1 in Dorais [6] for an explicit definition of Kleene's realizability. Troelstra [39] introduced an intuitionistic subsystem called $\mathsf{EL_0}$ that does not include the law of excluded middle. In fact,

Theorem 2.1.7 (Troelstra [39]).

$$\mathsf{RCA}_0 = \mathsf{EL}_0 + \mathsf{LEM}$$

So in a sense EL_0 is an intuitionistic analogue to RCA_0.

Hirst and Mummert [22] and Dorais [6] have applied Kleene's modified realizability to prove some general theorems about sequential forms, relating a statement's provability in intuitionistic subsystems to the sequential form's provability in classical subsystems. A major result of Dorais (Corollary 2.9 and Corollary 3.9 in [6]) is:

Theorem 2.1.8 (Dorais). *Suppose that $\alpha(X)$ and $\beta(X, Z)$ are formulas that satisfy Condition Set Γ defined below.*

(a) If

$$\mathsf{EL}_0 \vdash \forall X(\alpha(X) \to \exists Z\beta(X, Z))$$

then

$$\mathsf{RCA}_0 \vdash \forall X(\forall n\alpha(X_n) \to \exists Z\forall n\beta(X_n, Z_n)).$$

(b) If

$$\mathsf{EL}_0 + \mathsf{WKL} \vdash \forall X(\alpha(X) \to \exists Z\beta(X, Z))$$

then

$$\mathsf{WKL}_0 + \mathsf{B}\Sigma^0_2 \vdash \forall X(\forall n\alpha(X_n) \to \exists Z\forall n\beta(X_n, Z_n)).$$

Taking the contrapositive, if a statement's sequential form is strictly stronger than RCA_0, then the original statement does not have a constructive proof, and every proof must necessarily make use of LEM. If a sequential form is equivalent to ACA_0 (which is strictly stronger than $\mathsf{WKL}_0 + \mathsf{B}\Sigma^0_2$), then even assuming WKL we cannot prove the original statement without using LEM. As mentioned above, there is a necessary condition for Theorem 2.1.8:

Condition Set Γ. The following are conditions on $\alpha(X)$ and $\beta(X, Z)$ for Theorem 2.1.8 to hold. For part (a), $\alpha(X)$ must belong to the syntactic class N_K and $\beta(X, Z)$ must belong to the syntactic class Γ_K. For part (b), $\alpha(X)$ must belong to the syntactic class N_L and $\beta(X, Z)$ must belong to the syntactic class Γ_L. We will not define these four syntactic classes here, but definitions can be found in Dorais [6].

In all applications considered in this thesis, the statements $\alpha(X)$ and $\beta(X, Z)$ collectively assert that for every finite sequence of rational numbers $\bar{a}$ there exists a finite sequence of rational numbers $\bar{b}$ such that some computable relation holds: $\bar{a}\ R\ \bar{b}$. (So syntactically, the relation R can be written both in a Σ^0_1 form and in a Π^0_1 form.) In these cases, $\alpha(X)$ and $\beta(X, Z)$ are well within the syntactic requirements for the above theorem to hold. See Definition 3.1.4 for the most common formulation of our sequential problems.

Some results about sequential forms proved in Dorais, Hirst and Shafer [9], Fujiwara and Yokoyama [12], Hirst [21], and Hirst and Mummert [22] include:

Theorem 2.1.9.

- $\mathsf{WKL}_0 \leftrightarrow$ *Sequential Dichotomy of Reals: If $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ is a sequence of reals, there is a set $I \subseteq \mathbb{N}$ such that $\forall i\,[(i \in I \rightarrow \alpha_i \geq 0)\ \wedge\ (i \notin I \rightarrow \alpha_i \leq 0)]$. [9]*

- $\mathsf{ACA}_0 \leftrightarrow$ *Sequential Trichotomy of Reals: If $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ is a sequence of nonnegative reals, there is a set $I \subseteq \mathbb{N}$ such that $\forall i\,(i \in I \leftrightarrow \alpha_i > 0)$. [9]*

- $\mathsf{WKL}_0 \leftrightarrow$ *For every sequence $\langle X_n \rangle_{n \in \mathbb{N}}$ of quickly converging Cauchy sequences there is a sequence $\langle Y_n \rangle_{n \in \mathbb{N}}$ of Dedekind cuts such that X_i is equivalent to Y_i for all $i \in \mathbb{N}$. [21]*

- $\mathsf{ACA}_0 \leftrightarrow$ *For every sequence* $\langle M_n \rangle_{n \in \mathbb{N}}$ *of 2 × 2 real matrices, such that each matrix* M_n *has only real eigenvalues, there are sequences* $\langle U_n, J_n \rangle_{n \in \mathbb{N}}$ *such that* (U_i, J_i) *is a Jordan decomposition of* M_i *for all* $i \in \mathbb{N}$. *[22]*

- $\mathsf{WKL}_0 \leftrightarrow$ *For every sequence* $\langle f_n \rangle_{n \in \mathbb{N}}$ *of continuous functions* $f_n : [-1, 1] \to \mathbb{R}$, *there exists a sequence of maxima* $\langle a_n \rangle_{n \in \mathbb{N}}$, $a_n \in [-1, 1]$, *such that* $f_n(a_n) = \max\limits_{x \in [-1,1]} f_n(x)$. *[12]*

- $\mathsf{ACA}_0 \leftrightarrow$ *For every sequence* $\langle f_n \rangle_{n \in \mathbb{N}}$ *of continuous functions* $f_n : (-1, 1) \to \mathbb{R}$ *such that* $f_n(0) > 0$ *and* $\lim\limits_{x \to \pm 1} f_n(x) = 0$, *there exists a sequence of maxima* $\langle a_n \rangle_{n \in \mathbb{N}}$, $a_n \in (-1, 1)$, *such that* $f_n(a_n) = \max\limits_{x \in (-1,1)} f_n(x)$. *[12]*

Dorais [6] and Hirst and Mummert [22] use the above examples to show that none of the corresponding nonsequential statements can be proved without the use of LEM.

2.2 Two Easy Examples

The two theorems in this section will be generalized in Chapter 3, and so technically the reader can skip this section and not miss any results. However, I include these examples to show the basic technique of proving the strength of a sequential problem.

Theorem 2.2.1. *The following are equivalent over* RCA_0*:*

(i) ACA_0

(ii) The Sequential Finite Pigeonhole Principle: Given $k \geq 2$ *and a sequence* $\langle A_n, f_n \rangle_{n \in \mathbb{N}}$, *where* A_n *is a finite set and* $f_n : A_n \to \{0, 1, \ldots, k-1\}$, *there is a sequence* $\langle y_n \rangle_{n \in \mathbb{N}}$ *such that*

$$\forall n \left(|\{x \in A_n : f_n(x) = y_n\}| \geq \frac{|A_n|}{k} \right).$$

Proof. (i) implies (ii) by Proposition 2.1.4. Assume (ii). Let A be an oracle. We will show that the halting set relative to A exists, implying ACA_0.

Define $\langle f_n \rangle_{n \in \mathbb{N}}$ as follows: We always have

$$\begin{aligned} f_n(0) &= 0 & & \\ f_n(1) &= 1 & f_n(2) &= 1, \\ f_n(3) &= 2 & f_n(4) &= 2, \\ &\dots & & \\ f_n(2k-3) &= k-1 & f_n(2k-2) &= k-1. \end{aligned}$$

If $\Phi_n^A(n) \downarrow [s]$ and s is the first stage at which it halts, then we also have

$$f_n(2k+s) = 0 \qquad f_n(2k+s+1) = 0.$$

Note that $\langle f_n \rangle_{n \in \mathbb{N}}$ is a computable sequence of functions.

Let $\langle y_n \rangle_{n \in \mathbb{N}}$ be the sequence guaranteed by (ii). Then $y_n = 0$ if and only if $\Phi_n^A(n)$ halts. For if $\Phi_n^A(n)$ halts, then

$$|\{x \in A_n : f_n(x) = 0\}| = 3 > 2 + \frac{1}{k} = \frac{2k+1}{k} = \frac{|A_n|}{k}.$$

If $j > 0$, then

$$|\{x \in A_n : f_n(x) = j\}| = 2 < 2 + \frac{1}{k} = \frac{2k+1}{k} = \frac{|A_n|}{k}.$$

Therefore, y_n must be 0.

On the other hand, if $\Phi_n^A(n)$ does not halt, then

$$|\{x \in A_n : f_n(x) = 0\}| = 1 < 2 - \frac{1}{k} = \frac{2k-1}{k} = \frac{|A_n|}{k}$$

and so we must have $y_n > 0$.

Therefore, $y_n = 0$ if and only if $\Phi_n^A(n)\downarrow$, and we can compute the Turing jump A', which proves $\mathsf{ACA_0}$.

$\square$

At first glance, the sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in the above proof may look noncomputable, since we are modifying it if and only if $\Phi_n(n)\downarrow$. However, we modify it if and only if $\Phi_n(n)\downarrow$ *at a particular stage* s, which is something we can compute. Our sequence of functions is computable, but the problem of finding a sequence of *complete lists* of ordered pairs in each f_n is noncomputable.

Theorem 2.2.2. *The following are equivalent over* $\mathsf{RCA_0}$*:*

(i) $\mathsf{WKL_0}$

(ii) If $\langle G_n \rangle_{n\in\mathbb{N}} = \langle V_n, E_n \rangle_{n\in\mathbb{N}}$ *is a sequence of finite graphs without odd cycles, then there is a sequence of bipartitions* $\langle j_n \rangle_{n\in\mathbb{N}}$, $j_n : V_n \to \{0,1\}$, *such that*

$$\forall n\, [\forall v\, \forall w\, (\{v,w\} \in E_n \;\to\; j_n(v) \neq j_n(w))\,].$$

Proof. (i) $\Rightarrow$ (ii): We can view $\langle G_n \rangle_{n\in\mathbb{N}}$ as an infinite graph $G = (V, E)$, with $(n, v) \in V$ if and only if $v \in V_n$, and $((n, v),\, (n', w)) \in E$ if and only if $n = n'$ and $(v, w) \in E_n$. By Hirst [20], $\mathsf{WKL_0}$ proves that an infinite graph G with no odd cycles is bipartite. Therefore, there exists $j : V \to \{0,1\}$ such that $((m, v), (n, w)) \in E \to j(m, v) \neq j(n, w)$. Define $j_n : V_n \to \{0,1\}$ by $j_n(v) = j(n, v)$. Then $(v, w) \in E_n$ implies

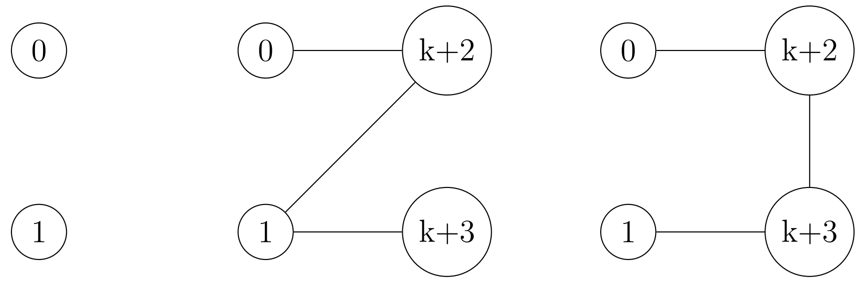

Figure 2.1: Possible bipartite graphs in the proof of Theorem 2.2.2.

that $((n, v), (n, w)) \in E$, which implies that $j(n, v) \neq j(n, w)$; i.e., $j_n(v) \neq j_n(w)$ as desired.

(ii) $\Rightarrow$ (i): We prove Σ^0_1-separation which is equivalent to $\mathsf{WKL_0}$; . Let f, g be injections with disjoint ranges. Define a sequence of graphs $\langle G_n \rangle_{n\in\mathbb{N}} = \langle V_n, E_n \rangle_{n\in\mathbb{N}}$ as follows:

0, 1 are always in V_n. If $f(k) = n$, then $k + 2$, $k + 3$ are in V_n and $(0, k + 2)$, $(k + 2, 1)$, and $(1, k + 3)$ are in E_n. If $g(k) = n$, then $k + 2$, $k + 3$ are in V_n and $(0, k + 2)$, $(k + 2, k + 3)$, and $(1, k + 3)$ are in E_n. V_n and E_n are always computable from f, g. See Figure 2.1.

Let $\langle j_n \rangle_{n\in\mathbb{N}}$ be the sequence of bipartitions given by (ii). Define $h : \mathbb{N} \to \{0, 1\}$ by $h(n) = 1$ if $j_n(0) = j_n(1)$, and $h(n) = 0$ otherwise. Then if n is in the range of f, $h(n) = 1$, and if n is in the range of g, $h(n) = 0$. If n is not in either range, then since 0, 1 are disconnected from each other, $h(0)$ and $h(1)$ could be either 0 or 1. It is clear that h gives a separating set for f and g, which proves $\mathsf{WKL_0}$.

□

If you assume that all G_n are *connected*, then the sequential bipartition theorem can be proved constructively in $\mathsf{RCA_0}$. Namely, given $a \in V_n$, to find $j_n(a)$, first find the smallest natural number in V_n, call it a_0, then enumerate V_n and E_n until we have a path from a_0 to a. If $a = a_0$, let $j_n(a) = 0$; otherwise, if the path has odd length let

$j_n(a) = 0$, and if the path has even length let $j_n(a) = 1$. Since G_n has no odd cycles, each j_n is well-defined and has the desired properties.

As a consequence of Theorem 2.1.8, we have:

Corollary 2.2.3. *The nonsequential versions of Theorems 2.2.1 and 2.2.2 are not consequences of* EL_0*, i.e., neither can be proved without the law of excluded middle. Moreover, the nonsequential Pigeonhole Principle (2.2.1) is not a consequence of* $\mathsf{EL}_0 + \mathsf{WKL}$.

2.3 The Principles $\mathsf{Predict}_k(r)$ and $\mathsf{Evade}_k(r)$

This section is joint work with Dorais.

Recall from Section 1.1 that DNR is the principle that asserts the existence of a diagonally nonrecursive function, i.e., a function $g : \mathbb{N} \to \mathbb{N}$ with $\forall e(g(e) \neq \Phi_e(e))$, avoiding every recursive function at at least one point. The related principle $\mathsf{DNR}(r)$ also asserts that the range of g is bounded by r:

Definition 2.3.1. Let $r \geq 2$. $\mathsf{DNR}(r)$ is the statement that for any oracle A, there exists $g : \mathbb{N} \to r$ such that

$$\forall e \left(g(e) \neq \Phi_e^A(e)\right)$$

While WKL_0 is stronger than DNR, it is equivalent to $\mathsf{DNR}(r)$ for standard r:

Theorem 2.3.2 (Jockusch and Soare [26]). *For* $2 \leq r < \omega$, $\mathsf{RCA}_0 \vdash (\mathsf{DNR}(r) \leftrightarrow \mathsf{WKL}_0)$.

In [8], Dorais, Hirst, and Shafer prove that in models where $\mathsf{I}\Sigma^0_2$ fails, Theorem 2.3.2 is not necessarily true when r is nonstandard:

Theorem 2.3.3 (Dorais, Hirst, and Shafer). $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \exists r \mathsf{DNR}(r) \nvdash \mathsf{WKL}_0$.

If we replace $\mathsf{B}\Sigma^0_2$ with $\mathsf{I}\Sigma^0_2$, then Theorem 2.3.3 is false. In any model of $\mathsf{RCA_0}+\mathsf{I}\Sigma^0_2$, $\exists r\mathsf{DNR}(r)$ is indeed equivalent to $\mathsf{WKL_0}$.

In our upcoming paper [7], Dorais and I introduce two new principles, $\mathsf{Predict}_k(r)$ and $\mathsf{Evade}_k(r)$. These principles have not been defined in the literature before, though Schmerl did use them implictly in [33], as we will see more clearly in Section 2.4. $\mathsf{Evade}_k(r)$ is related to $\mathsf{DNR}(r)$, and at first appears to be a weakening of $\mathsf{DNR}(r)$, but in most cases they are actually equivalent. $\mathsf{Predict}_k(r)$ is the negation of $\mathsf{Evade}_k(r)$.

Definition 2.3.4. $\mathsf{Predict}_k(r)$ is the following statement:

There is a sequence $\langle h_0, \ldots, h_{k-1}\rangle$ of partial $\mathbf{\Sigma}^0_1$-functions $h_i : D_{i+1} \to \{0, \ldots, r-1\}$ whose domains form a nested sequence of $\mathbf{\Sigma}^0_1$-sets

$$D_0 = \mathbb{N} \supseteq D_1 \supseteq \cdots \supseteq D_k$$

such that if $\langle f_0, \ldots, f_{k-1}\rangle$ is any sequence of partial $\mathbf{\Sigma}^0_1$-functions $f_i : D_i \to \{0, \ldots, r-1\}$ then there is an $x \in D_k$ such that $f_i(x) = h_i(x)$ for all $i < k$.

In other words, the h_i's can "predict" each set of f_i's one might choose, at at least one value. This is a bold statement, since we can choose the f_i's to be whatever we want, and in fact it is more than just bold. It is false in any model of full second-order arithmetic.

Notice that the $\mathbf{\Sigma}^0_1$ symbols are in boldface. This means that there is an oracle A and a sequence of partial $\Sigma^{0,A}_1$-functions $\langle h_0, \ldots, h_{k-1}\rangle$, such that for any oracle B and any sequence of partial $\Sigma^{0,B}_1$-functions $\langle f_0, \ldots, f_{k-1}\rangle$, the conclusion holds. Also observe that for each i, f_i is defined on a larger domain than h_i.

In Subsection 2.3.1, we will see that if any sequence of partial A-computable functions $\langle h_0, \ldots, h_{k-1}\rangle$ witnesses $\mathsf{Predict}_k(r)$, then there is a *particular sequence* $\langle \Delta^A_0, \ldots, \Delta^A_{k-1}\rangle$ that witnesses it. Most references to $\mathsf{Predict}_k(r)$ beyond this section make use of the Δ_i^A's. See Definition 2.3.14 and Theorem 2.3.15.

Definition 2.3.5. $\mathsf{Evade}_k(r)$ is the negation of $\mathsf{Predict}_k(r)$: Given any sequence $\langle h_0, \ldots, h_{k-1} \rangle$ of partial $\mathbf{\Sigma}_1^0$-functions $h_i : D_{i+1} \to \{0, \ldots, r-1\}$ whose domains form a nested sequence

$$D_0 = \mathbb{N} \supseteq D_1 \supseteq \cdots \supseteq D_k,$$

there is a sequence $\langle f_0, \ldots, f_{k-1} \rangle$ of partial $\mathbf{\Sigma}_1^0$-functions $f_i : D_i \to \{0, \ldots, r-1\}$ such that for every $x \in D_k$ we have $f_i(x) \neq h_i(x)$ for some $i < k$.

$\mathsf{Evade}_k(r)$ asserts that for every such sequence $\langle h_0, \ldots, h_{k-1} \rangle$, there is a sequence $\langle f_0, \ldots, f_{k-1} \rangle$ such that each point in the common domain D_k "evades" *at least one* of the h_i's. This statement recalls the diagonal non-recursion property, and for $k = 1$, it is easy to see that they are equivalent:

Proposition 2.3.6 (RCA_0)**.** *For every* $r \geq 2$, $\mathsf{DNR}(r) \leftrightarrow \mathsf{Evade}_1(r)$.

Proof. To see that $\mathsf{Evade}_1(r)$ implies $\mathsf{DNR}(r)$, given an oracle A, apply $\mathsf{Evade}_1(r)$ to the partial $\Sigma_1^{0,A}$-function

$$h_0(x) = \begin{cases} \Phi_x^A(x) & \text{when } \Phi_x^A(x)\downarrow < r \\ \uparrow & \text{otherwise,} \end{cases}$$

to obtain a function $f_0 : \mathbb{N} \to \{0, \ldots, r-1\}$ such that $f_0(x) \neq \Phi_x^A(x)$ whenever $\Phi_x^A(x)\downarrow$.

To see that $\mathsf{DNR}(r)$ implies $\mathsf{Evade}_1(r)$, assume $\mathsf{DNR}(r)$ and suppose on the contrary that $h_0 : D_1 \to \{0, \ldots, r-1\}$ and $D_0 = \mathbb{N} \supseteq D_1$ witness $\mathsf{Predict}_1(r)$ (i.e., $\neg\mathsf{Evade}_1(r)$). Let A be an oracle such that h_0 is a partial A-computable function. By the relativized s-m-n theorem, there is a primitive recursive injection $s : \mathbb{N} \to \mathbb{N}$ such that $h_0(x) = \Phi_{s(x)}^A(y)$ for all $x, y \in \mathbb{N}$. If $f : \mathbb{N} \to \{0, \ldots, r-1\}$ is any function, then by $\mathsf{Predict}_1(r)$

there is an $x \in D_1$ such that $f(s(x)) = h_0(x) = \Phi^A_{s(x)}(s(x))$. Thus, $s(x)$ witnesses that f does not satisfy $\mathsf{DNR}(r)$ relative to A. $\square$

It is easy to see that if $k \leq \ell$ then $\mathsf{Evade}_k(r)$ implies $\mathsf{Evade}_\ell(r)$ and, contrapositively, $\mathsf{Predict}_\ell(r)$ implies $\mathsf{Predict}_k(r)$. Similarly, if $r \leq s$ then $\mathsf{Evade}_k(r)$ implies $\mathsf{Evade}_k(s)$ and, contrapositively, $\mathsf{Predict}_k(s)$ implies $\mathsf{Predict}_k(r)$. We have thus seen that for all k and r, $1 \leq k < \omega$, $2 \leq r < \omega$, RCA_0 proves that

$$\mathsf{WKL}_0 \leftrightarrow \mathsf{DNR}(r) \leftrightarrow \mathsf{Evade}_1(r) \rightarrow \mathsf{Evade}_k(r)$$

so we clearly have that $\mathsf{Evade}_k(r)$ is true, and $\mathsf{Predict}_k(r)$ is false, in any model of full second-order arithmetic.

For standard k, the reverse implication is also true: $\mathsf{Evade}_k(r) \rightarrow \mathsf{Evade}_1(r)$. The next two propositions will help us prove this.

Proposition 2.3.7 (RCA_0)**.** *For all $k, \ell, r \geq 1$, $\mathsf{Predict}_{k\ell}(r)$ implies $\mathsf{Predict}_k(r^\ell)$. Contrapositively, $\mathsf{Evade}_k(r^\ell)$ implies $\mathsf{Evade}_{k\ell}(r)$.*

Proof. Suppose $\langle h_0, \ldots, h_{k\ell-1}\rangle$ is a sequence of partial $\mathbf{\Sigma}^0_1$-functions $h_j : D_{j+1} \rightarrow \{0, \ldots, r-1\}$ that witness $\mathsf{Predict}_{k\ell}(r)$. For $i \leq k$, let $D'_i = D_{i\ell}$ and let $h'_i : D'_{i+1} \rightarrow \{0, \ldots, r^\ell - 1\}$ be defined by

$$h'_i(x) = \sum_{j=0}^{\ell-1} h_{i\ell+j}(x) r^j.$$

Given a sequence $\langle f'_0, \ldots, f'_{k-1}\rangle$ of partial $\mathbf{\Sigma}^0_1$-functions $f'_i : D'_i \rightarrow \{0, \ldots, r^\ell - 1\}$, define $f_j : D_j \rightarrow \{0, \ldots, r-1\}$ for $j < k\ell$ in such a way that if $i < k$ and $m < \ell$, then

$$\sum_{j=0}^{m} f_{i\ell+j}(x) r^j \equiv f'_i(x) \pmod{r^{m+1}}$$

for $x \in D_{i\ell+m}$. If $x \in D'_k = D_{k\ell}$ is such that $f_j(x) = h_j(x)$ for all $j < k\ell$, then also $f'_i(x) = h'_i(x)$ for all $i < k$. $\square$

Corollary 2.3.8 ($\mathsf{RCA_0}$)**.** *For all k and r, $\mathsf{DNR}(r^k) \to \mathsf{Evade}_k(r)$.*

Proof. This follows from Propositions 2.3.6 and 2.3.7.

$\square$

Proposition 2.3.9 ($\mathsf{RCA_0}$)**.** *For all k and r, $\mathsf{Predict}_k(r)$ implies $\mathsf{Predict}_{2k}(r)$. Contrapositively, $\mathsf{Evade}_{2k}(r)$ implies $\mathsf{Evade}_k(r)$.*

Proof. Let $\langle h_0, \ldots, h_{k-1}\rangle$ and $D_0, \ldots, D_k$ be as in $\mathsf{Predict}_k(r)$. For $i < k$, define

$$D'_{i+1} = \{\langle x, y\rangle : x \in D_{i+1}\}, \quad D'_{k+i} = \{\langle x, y\rangle : x \in D_k, y \in D_i\},$$

and

$$h'_i(\langle x, y\rangle) = h_i(x), \quad h'_{k+i}(\langle x, y\rangle) = h_i(y).$$

We claim that the sequence $h'_0, \ldots, h'_{2k-1}$ witnesses $\mathsf{Predict}_{2k}(r)$.

Suppose $\langle f'_0, \ldots, f'_{2k-1}\rangle$ is a sequence of partial $\mathbf{\Sigma}^0_1$-functions $f'_i : D'_i \to \{0, \ldots, r-1\}$. Given any $y \in \mathbb{N}$, it follows from $\mathsf{Predict}_k(r)$ that there is some $x \in D_k$ such that $f'_i(\langle x, y\rangle) = h_i(x) = h'_i(\langle x, y\rangle)$ for all $i < k$. Represent this relationship with a function $f : \mathbb{N} \to D_k$ such that $f(y) = x$; this function exists in our model. Consider the functions $f_i : D_i \to \{0, \ldots, r-1\}$ defined by $f_i(y) = f'_{k+i}(\langle f(y), y\rangle)$. It follows from $\mathsf{Predict}_k(r)$ that there is some $y_0 \in D_k$ such that $f_i(y_0) = h_i(y_0)$ for $i < k$. We then have

$$f'_i(\langle f(y_0), y_0\rangle) = h_i(f(y_0)) = h'_i(\langle f(y_0), y_0\rangle)$$

and

$$f'_{k+i}(\langle f(y_0), y_0\rangle) = f_i(y_0) = h_i(y_0) = h'_{k+i}(\langle f(y_0), y_0\rangle)$$

for all $i < k$. $\square$

Corollary 2.3.10 (RCA_0)**.** *For all k and r, $\mathsf{Predict}_k(r)$ implies $\mathsf{Predict}_k(r^2)$ and $\mathsf{Evade}_k(r^2)$ implies $\mathsf{Evade}_k(r)$.*

Proof. This follows from Propositions 2.3.7 and 2.3.9.

$\square$

Remark 2.3.11. In the proofs of both Proposition 2.3.7 and Proposition 2.3.9, we construct new witnessing functions that are $\Sigma_1^{0,A}$ relative to the same oracle A as our initial functions. By this we mean: if the given functions in Proposition 2.3.7 that witness $\mathsf{Predict}_{k\ell}(r)$ are $\Sigma_1^{0,A}$-functions, then the functions we construct that witness $\mathsf{Predict}_k(r^\ell)$ are also $\Sigma_1^{0,A}$; in Proposition 2.3.9, if the given functions that witness $\mathsf{Predict}_k(r)$ are $\Sigma_1^{0,A}$-functions, then the functions we construct that witness $\mathsf{Predict}_{2k}(r)$ are also $\Sigma_1^{0,A}$.

Corollary 2.3.12 (RCA_0)**.** *For every k, r, $1 \leq k < \omega$, $2 \leq r < \omega$, $\mathsf{Predict}_1(2)$ implies $\mathsf{Predict}_k(r)$.*

Proof. This follows from repeated application of Proposition 2.3.9 and Corollary 2.3.10.

$\square$

And finally we have a significant corollary about the strength of $\mathsf{Evade}_k(r)$ for every standard k, r:

Corollary 2.3.13 (RCA_0)**.** *For every k, r, $1 \leq k < \omega$, $2 \leq r < \omega$, $\mathsf{Evade}_k(r) \leftrightarrow \mathsf{WKL}_0$.*

Proof. Collectively, Theorem 2.3.2, Proposition 2.3.6, and Corollary 2.3.12 state that $\mathsf{WKL}_0 \leftrightarrow \mathsf{DNR}(2) \leftrightarrow \mathsf{Evade}_1(2) \leftrightarrow \mathsf{Evade}_k(r)$.

$\square$

2.3.1 Existence of canonical witnesses

We now show, by an argument due to Dorais, that for every oracle A and parameters k, r there is a *canonical* choice of partial A-computable functions $\langle h_0, \ldots, h_{k-1}\rangle$ that witness $\mathsf{Predict}_k(r)$ if any such functions witness $\mathsf{Predict}_k(r)$. We will call this sequence $\langle \Delta_0^A, \ldots, \Delta_{k-1}^A\rangle$. This will be useful to us, for if we wish to show that $\mathsf{Evade}_k(r)$ holds, it suffices to show that no $\langle f_0, \ldots, f_{k-1}\rangle$ sequence exists for this particular sequence of Δ_i^A's.

Definition 2.3.14 (RCA_0)**.** Let p_i be the $(i+1)^{\text{st}}$ prime number and, let $\nu_i(x)$ denote the exponent of p_i in the prime factorization of $x+1$. Given an oracle A, define

$$\Delta_i^A(x) = \begin{cases} \Phi^A_{\nu_i(x)}(x) & \text{if } \Phi^A_{\nu_j(x)}(x)\downarrow \text{ for all } j \leq i, \\ \uparrow & \text{otherwise.} \end{cases}$$

Also, let $U_{i+1}^A = \mathrm{dom}(\Delta_i^A)$ and $U_0^A = \mathbb{N}$.

Proposition 2.3.15 (Dorais, RCA_0)**.** *For all k, r and every set A, if $\mathsf{Predict}_k(r)$ is witnessed by some sequence of partial A-computable functions, then $\Delta_0^A, \ldots, \Delta_{k-1}^A$ witness $\mathsf{Predict}_k(r)$. More precisely, it is the restrictions of $\Delta_0^A, \ldots, \Delta_{k-1}^A$ to the inverse image of $\{0, \ldots, r-1\}$ that witness $\mathsf{Predict}_k(r)$.*

The second statement is technically necessary, since $\mathsf{Predict}_k(r)$ asserts the existence of a sequence of partial $\mathbf{\Sigma}_1^0$-functions with range $\{0, \ldots, r-1\}$.

Proof. Suppose $h_i : D_{i+1} \to \{0, \ldots, r-1\}$, $i < k$, is a sequence of partial A-computable functions that witness $\mathsf{Predict}_k(r)$. By the relativized s-m-n theorem, there are primitive recursive injections $s_0, \ldots, s_{k-1}$ such that $h_i(x) = \Phi^A_{s_i(x)}(y)$ for all $x, y \in \mathbb{N}$. Define $s(x) = p_0^{s_0(x)} \cdots p_{k-1}^{s_{k-1}(x)} - 1$, which is also a primitive recursive injection. Note that since

$$\mathbb{N} = D_0 \supseteq D_1 \supseteq \cdots \supseteq D_k,$$

we then have

$$\Delta_i^A(s(x)) = \Phi_{s_i(x)}^A(s(x)) = h_i(x)$$

for all $x \in D_{i+1}$ and $\Delta_i^A(s(x))\uparrow$ when $x \notin D_{i+1}$. In particular, $s(x) \in U_{i+1}^A$ iff $x \in D_{i+1}$.

Suppose $f_i : U_i^A \to \{0, \dots, r-1\}$, $i < k$, is a sequence of partial $\mathbf{\Sigma}_1^0$-functions. Then, the composite functions $f_i'(x) = f_i(s(x))$ form a sequence of partial $\mathbf{\Sigma}_1^0$-functions $f_i' : D_i \to \{0, \dots, r-1\}$. Therefore, by hypothesis that $h_0, \dots, h_{k-1}$ witness $\mathsf{Predict}_k(r)$, there is an $x \in D_k$ such that

$$f_i(s(x)) = f_i'(x) = h_i(x) = \Delta_i^A(s(x))$$

for all $i < k$. It follows that $\Delta_0^A, \dots, \Delta_{k-1}^A$ also witness $\mathsf{Predict}_k(n)$. $\square$

Theorem 2.3.16 (RCA_0+$\mathsf{I}\Sigma_2^0$). *For every* $r \geq 1$, $\mathsf{Predict}_1(r)$ *implies* $(\forall k \geq 1)\mathsf{Predict}_k(r)$.

Proof. Fix $r \in \mathbb{N}$, and assume that $\mathsf{Predict}_1(r)$ holds. By Proposition 2.3.15, there is an oracle A such that $\langle \Delta_0^A \rangle$ witnesses $\mathsf{Predict}_1(r)$. Let $k \in \mathbb{N}$ be such that $\mathsf{Predict}_k(r)$ fails, and so certainly $\langle \Delta_0^A, \dots, \Delta_{k-1}^A \rangle$ does not satisfy the hypotheses of $\mathsf{Predict}_k(r)$. Let $\langle f_0, \dots, f_{k-1} \rangle$ be a $\mathbf{\Sigma}_1^0$-sequence of functions witnessing this failure of $\mathsf{Predict}_k(r)$ in the $\langle \Delta_i^A \rangle_{i<k}$ sequence. So $f_i : U_i \to \{0, \dots, r-1\}$, and for each $x \in U_k^A$ there is $i < k$ such that $f_i(x) \neq \Delta_i^A(x)$.

For $j \in \mathbb{N}$, define $\Theta(j)$ as follows: There exists $i \geq j$ such that for all $x \in U_{k-i}^A$, we have

$$\big(f_0(x), f_1(x), \dots f_{k-1-i}(x)\big) \neq \big(\Delta_0^A(x), \Delta_1^A(x), \dots, \Delta_{k-1-i}^A(x)\big)$$

Notice that $\Theta(j)$ is a Σ^0_2-statement. By hypothesis, we have $\Theta(0)$ is true. However, $\Theta(k-1)$ must be false, since this would mean that for all $x \in U_1^A$, we have $f_0(x) \neq \Delta_0^A(x)$, which would contradict that $\langle \Delta_0^A \rangle$ witnesses $\mathsf{Predict}_1(r)$.

By $\mathsf{I}\Sigma^0_2$, since $\forall j \Theta(j)$ fails and since $\Theta(0)$ holds, there exists j_0 such that $\Theta(j_0) \wedge \neg\Theta(j_0+1)$ holds.

Then the sequence $\langle f_0, \ldots, f_{j_0+1} \rangle$ does witness that $\mathsf{Predict}_{j_0+2}(r)$ fails for $\langle \Delta_0^A, \ldots, \Delta_{j_0+1}^A \rangle$, but $\langle f_0, \ldots, f_{j_0} \rangle$ does not witness the failure of $\mathsf{Predict}_{j_0+1}$ for $\langle \Delta_0^A, \ldots, \Delta_{j_0}^A \rangle$.

Thus we have that $\Delta_{j_0+1}^A : U_{j_0+2} \to \{0, \ldots, r-1\}$, $f_{j_0+1} : U_{j_0+1} \to \{0, \ldots, r-1\}$, and for any $x \in U_{j_0+2}$, $f_{j_0+1}(x) \neq \Delta_{j_0+1}^A(x)$. We claim that this witnesses the failure of $\mathsf{Predict}_1(r)$.

To prove our claim, we revisit the proof of Proposition 2.3.15; recall the formal definitions of the Δ_i^A's from this proof. Suppose $h : D_1 \to \{0, \ldots, r-1\}$ witnesses $\mathsf{Predict}_1(r)$. By the relativized s-m-n theorem, there is a primitive recursive injection s_0 such that $h(x) = \Phi^A_{s_0(x)}(y)$ for all $x, y \in \mathbb{N}$. Define $s(x) = p_0 \cdot p_1 \cdots p_{j_0} \cdot p_{j_0+1}^{s_0(x)} - 1$. Then $\Delta_{j_0+1}^A(s(x)) = \Phi^A_{s_0(x)}(s(x)) = h(x)$. In particular, $s(x) \in U_{j_0+2}$ iff $x \in D_1$.

The composite function $g(x) = f_{j_0+1}(s(x))$ is a function $g : \mathbb{N} \to \{0, \ldots, r-1\}$ and we have shown that for any $x \in U_{j_0+2}$ we have $f_{j_0+1}(x) \neq \Delta_{j_0+1}^A(x)$. Let $x \in D_1$ be arbitrary; then $s(x) \in U_{j_0+2}$ and

$$g(x) = f_{j_0+1}(s(x)) \neq \Delta_{j_0+1}^A(s(x)) = h(x)$$

hence we have shown that $\mathsf{Predict}_1(r)$ fails, a contradiction. So our original assumption was wrong, and $\mathsf{Predict}_k(r)$ holds. This completes the proof.

$\square$

Theorem 2.3.17 (RCA_0+$\mathsf{I}\Sigma^0_2$)**.** *For every* $k \geq 1$, $\mathsf{Predict}_k(2)$ *implies* $(\forall r \geq 1)\mathsf{Predict}_k(r)$.

Proof. Assume $\mathsf{Predict}_k(2)$. If $\mathsf{Predict}_k(2^r)$ fails, then $\mathsf{Predict}_{kr}(2)$ fails by Corollary

2.3.8, and so $\mathsf{Predict}_1(2)$ fails by Theorem 2.3.16, and so clearly $\mathsf{Predict}_k(2)$ fails, a contradiction.

$\square$

Corollary 2.3.18 ($\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$)**.** $\mathsf{Predict}_1(2)$ *implies* $(\forall k, r \geq 1)\mathsf{Predict}_k(r)$.

Proof. This follows from Theorem 2.3.16 and Theorem 2.3.17.

$\square$

Corollary 2.3.19. *The following are each equivalent to* WKL_0 *over* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$:

- $\mathsf{Evade}_1(2)$ *or, equivalently,* $\mathsf{DNR}(2)$.
- $(\exists r \geq 1)\mathsf{Evade}_1(r)$ *or, equivalently,* $(\exists r \geq 1)\mathsf{DNR}(r)$.
- $(\exists k \geq 1)\mathsf{Evade}_k(2)$
- $(\exists k, r \geq 1)\mathsf{Evade}_k(r)$

We should not be surprised that $\mathsf{I}\Sigma^0_2$ is necessary for the last three statements. As mentioned earlier, Dorais, Hirst, and Shafer [8] showed that $\exists r\mathsf{DNR}(r)$ is strictly weaker than WKL_0, even if we assume $\mathsf{B}\Sigma^0_2$. Since $\exists r\, \mathsf{Evade}_1(r)$ implies $\exists r\, \mathsf{Evade}_1(2^r)$, which in turn implies $\exists r\, \mathsf{Evade}_r(2)$ by a uniform application of Proposition 2.3.7, we know that $\exists k\, \mathsf{Evade}_k(2)$ is also strictly weaker than WKL_0. The reverse implication is less clear, and its negation seems plausible:

Conjecture 2.3.20.

$$\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \exists k\mathsf{Evade}_k(2) \nvdash \exists r\mathsf{DNR}(r)$$

$$
\begin{tikzcd}
\mathsf{WKL}_0 \arrow[r, Leftrightarrow] & \mathsf{Evade}_k(r) \arrow[r, Leftrightarrow] & \mathsf{DNR}(r) \arrow[d, Rightarrow] \\
& & \exists r\mathsf{DNR}(r) \arrow[d, "Strict?"] \\
& & \exists k\mathsf{Evade}_k(2) \arrow[d, Rightarrow] \\
& & \mathsf{DNR} \arrow[d, Rightarrow] \\
& & \mathsf{RCA}_0
\end{tikzcd}
$$

Figure 2.2: Relationships between prediction principles in models of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Optimality of Friedberg's Lemma

Friedberg's Lemma, introduced in Jockusch [24], states that every r^2-bounded DNR function computes an r-bounded DNR function, and that this can be proven in RCA_0. Of course, we can apply the lemma repeatedly and conclude that every r^k-bounded DNR function computes an r-bounded DNR function for every $k \geq 1$. Dorais has proven that this is *optimal*: If $\ell(k,r)$ and $s(k,r)$ are primitive recursive functions such that

$$\mathsf{RCA}_0 \vdash (\forall k, r \geq 1)\,(\mathsf{Predict}_k(r) \to \mathsf{Predict}_{\ell(k,r)}(s(k,r)))$$

then we have

$$\frac{\ell(k,r)\log s(k,r)}{k\log r}$$

is bounded. Fixing $\ell(k,r) = k$ for the moment gives that $s(k,r) \leq r^b$ for some bound b.

The proof of this will be in our upcoming paper [7], but will not be included here. Let us instead turn to another application of the prediction principles: the analysis of colorability of different classes of graphs.

2.4 Prediction and On-Line Graph Colorability

One well-known theorem in computable graph theory is due to Gasarch and Hirst [13]: $\mathsf{WKL_0}$ is equivalent to the statement that every locally r-colorable graph is r-colorable. It follows that there exists a computable graph that is locally r-colorable, but which does not have a computable r-coloring.

Definition 2.4.1. Let $r \geq 2$. A graph $G = (V, E)$ is *locally r-colorable* if for every finite $V_0 \subseteq V$, the induced subgraph $(V_0, E \cap (V_0 \times V_0))$ is r-colorable.

Schmerl improved this result, showing that we can replace the two color parameters with any fixed standard ℓ, r.

Theorem 2.4.2 (Schmerl [32])**.** *Fix ℓ, r with $2 \leq \ell \leq r < \omega$. Then $\mathsf{WKL_0}$ is equivalent to the statement that every locally ℓ-colorable graph is r-colorable.*

So in models of $\neg\mathsf{WKL_0}$, there exist graphs that are locally ℓ-colorable but not r-colorable for arbitrarily high $r < \omega$.

In his subsequent paper [33], Schmerl considered which *classes* of graphs are computably colorable. Certainly one can imagine classes of graphs, such as the class of (connected) trees, which are provably colorable in $\mathsf{RCA_0}$. But is there a wide variety of classes of graphs, each of which contains a locally ℓ-colorable graph that is not globally r-colorable? Schmerl's paper [33] brought together the main ideas of this thesis for the first time, for the answer to this question depends on the on-line r-colorability of the class of graphs.

The definition of an on-line r-colorable class of graphs was given in Section 1.2; recall that it means that Bob (Player 2, the colorer) has a winning strategy in the game $\mathsf{G}(\mathcal{C}, r)$. Recall also that a universal class $\mathcal{C}$ of graphs is closed under isomorphisms and is such that $G \in \mathcal{C}$ if and only if every finite induced subgraph of G is in $\mathcal{C}$. A *natural* class of graphs is also closed under disjoint sums.

Definition 2.4.3. Let $\mathcal{C}$ be a class of graphs, and $p < \omega$. Let $\mathcal{C}^{(p)}$ be the subclass of $\mathcal{C}$ consisting of all graphs G in $\mathcal{C}$ all of whose components have no more than p vertices.

Here is Schmerl's significant result relating a graph's on-line colorability to its global colorability in the absence of $\mathsf{WKL_0}$:

Theorem 2.4.4 (Schmerl, Theorem 2.1 in [33])**.** *Let $r < \omega$. Let $\mathcal{C}$ be a natural class of graphs with a primitive recursive definition, and suppose that $\mathcal{C}$ is not on-line r-colorable. Then there is $p < \omega$ such that the following is provable in $\mathsf{RCA_0} + \neg\mathsf{WKL_0}$: There is a graph G in $\mathcal{C}^{(p)}$ that is not r-colorable.*

An alternative way to state the hypothesis is: "Suppose that there is $r < \omega$ such that Alice has a winning strategy in $\mathsf{G}(\mathcal{C}, r)$."

Recall that $\neg\mathsf{WKL_0} \leftrightarrow \mathsf{Predict}_k(r)$ for $1 \le k < \omega$, $2 \le r < \omega$. To prove Theorem 2.4.4, Schmerl makes use of $\mathsf{Predict}_k(r)$, though he did not call it that by name, after proving that $\mathsf{Predict}_k(r)$ holds in models of $\mathsf{RCA_0} + \neg\mathsf{WKL_0}$. The value of k here is the number of vertices that are required in the game $\mathsf{G}(\mathcal{C}, r)$ before Alice is guaranteed to defeat any play by Bob. This value k is also equal to p in the statement of the theorem, for all components of Alice's winning graph will have size at most $p = k$.

We reproduce Schmerl's proof below, with some details and notations slightly changed. First, however, we must show that the k $(= p)$ from the previous paragraph exists, and so we prove a couple of lemmas on the determinacy of the game.

Lemma 2.4.5 (Schmerl, essentially Lemma 1.1 in [33],)**.** *Let $\mathcal{C}$ be a natural class of graphs with a primitive recursive definition, and let $r < \omega$. If Bob has a winning strategy in $\mathsf{G}_k(\mathcal{C}, r)$ for every $k < \omega$, then Bob has a winning strategy in $\mathsf{G}(\mathcal{C}, r)$.*

Proof. We will define a tree T of winning strategies for Bob. Each node of T on level j will encode Bob's response to *every possible* sequence of plays by Alice, where Alice's j^{th} play is a binary sequence $< 2^j$ that encodes whether Alice's j^{th} vertex is connected to each of Alice's $j-1$ previous vertices. (The actual vertex played is not needed in this encoding.)

If $\sigma \in T$, then $\sigma(j)$ is a sequence $\langle c_{j,0},\, c_{j,1},\, \ldots,\, c_{j,N-1} \rangle$, such that N is the number of possible sequences of Alice's plays in rounds 0 through j, and $c_{j,i} \in (\, \{0, \ldots, r-1\}\,)^j$ is Bob's valid sequence of colors (i.e., the resulting coloring is proper) to Alice's i^{th} sequence of vertices and edge relations, in some canonical ordering. This is, of course, assuming that when Alice played the first $j' < j$ plays in her i^{th} sequence, Bob played the corresponding $c_{j',i'}$ encoded in $\sigma(j')$, which is $\sigma(j)$'s predecessor on level j' of the tree.

If for even one i, $0 \leq i < N$, Bob does not have a winning response, then the tree has a dead end, and $\sigma(j)$ is undefined.

By assumption, Bob has a winning strategy in $\mathsf{G}(\mathcal{C}, r)$ for every j. This means that T is infinite, and an infinite branch of T would clearly encode a full winning strategy for Bob in $\mathsf{G}(\mathcal{C}, r)$.

The tree is also bounded. Suppose that $\sigma(j)$ is a sequence of length N and its code has value M. I claim that in the recursive function $h(j, r)$ defined below, $h(j, 0)$ is a bound for N (number of possible plays for Alice in the first j steps) and $h(j, 1)$ is a bound for M (coded value of the sequence of possible responses by Bob in round j):

$$
\begin{aligned}
h(0,0) &= 2^0 \\
h(0,1) &= r^{h(0,0)} \\
h(1,0) &= h(0,0)\cdot 2^1 \\
h(1,1) &= r^{h(1,0)} \\
&\cdots \\
h(j,0) &= h(j-1,0)\cdot 2^j \\
h(j,1) &= r^{h(j,0)}
\end{aligned}
$$

Weak König's Lemma shows that this tree has an infinite path, and this path will encode a winning strategy for Bob in $\mathsf{G}(\mathcal{C}, r)$.

$\square$

Note: We could have proved the above lemma in $\mathsf{WKL_0}$, except that in our assumptions, we would have to replace "for every $k < \omega$" with "for every $k \in \mathbb{N}$." In the proofs of Corollary 2.4.7 and Theorem 2.4.4, it is the former that will be required.

Lemma 2.4.6 ($\mathsf{RCA_0}$)**.** *Let $\mathcal{C}$ be a natural class of graphs with a primitive recursive definition, and let $k, r \in \mathbb{N}$. Then $\mathsf{G}_k(\mathcal{C}, r)$ is determined.*

Proof. Since Alice's plays (edge relations) and Bob's plays (colorings) are bounded—specifically, their respective j^{th} plays are bounded by 2^j and r—Bob's full tree of strategies is bounded and computable. (Alice's actual set of vertex choices is *not* bounded, but that is not necessary for this proof.)

So we can perform a search for a winning strategy for Bob through the whole tree, and we will have an answer in finite time. If we do not come up with a winning strategy, then Alice has a winning strategy. Namely, play a_0 such that Bob does not have a winning strategy above a_0, and in general play a_i such that Bob does not have a winning strategy above a_i. We can computably find this sequence $\bar{a}$ of plays, and $\bar{a}$

will hence be a winning play for Alice.

□

Corollary 2.4.7. *Let $\mathcal{C}$ be a natural class of graphs with a primitive recursive definition, and let $r < \omega$. Then $\mathsf{G}(\mathcal{C}, r)$ is determined. In fact, if Bob does not have a winning strategy in $\mathsf{G}(\mathsf{C}, r)$, then there exists $k < \omega$ such that Alice has a winning strategy in $\mathsf{G}_k(\mathsf{C}, r)$.*

Proof. If Bob does not have a winning strategy in $\mathsf{G}(\mathcal{C}, r)$, then Lemma 2.4.5 implies that there exists $k \in \mathbb{N}$ such that Bob does not have a winning strategy in $\mathsf{G}_k(\mathcal{C}, r)$. Then Lemma 2.4.6 implies that Alice has a winning strategy in $\mathsf{G}_k(\mathcal{C}, r)$, and hence in $\mathsf{G}_k(\mathcal{C}, r)$.

□

Again, we could prove the above corollary in $\mathsf{WKL_0}$, except that we would have to replace "there exists $k < \omega$" with "there exists $k \in \mathbb{N}$."

We are ready to present Schmerl's proof of Theorem 2.4.4. We will generalize this proof in Theorem 3.2.5, which will look beyond graph colorings and prove a similar statement about any on-line problem.

Proof of Theorem 2.4.4. We construct an infinite graph with finite components, $G = (\mathbb{N} \times \mathbb{N}, E)$, such that $(n, x)\ E\ (n', y) \Rightarrow n = n'$. Let $G_n = \{(n, x) : x \in \mathbb{N}\}$; the previous statement shows that G_n and $G_{n'}$ are disconnected from each other, $n \neq n'$. Let $k < \omega$ be such that Alice has a winning strategy in k rounds of the game $\mathsf{G}(\mathcal{C}, r)$; such a k exists by Corollary 2.4.7. We will construct G so that $G_n = H_n \cup I_n$, where H_n is a graph of size at most k, and I_n is a graph with no edges, so that the full graph G is in the class $\mathcal{C}^{(k)}$.

Assume $\mathsf{RCA_0} + \neg\mathsf{WKL_0}$, which means that $\mathsf{Predict}_k(r)$ holds. We will define each graph G_n in terms of the canonical predictors $\langle \Delta_i^A \rangle_{i<k}$ defined in Definition 2.3.14.

Assume without loss of generality that if $\Delta^A_{i+1}(n) \downarrow [s]$, then $\Delta^A_i(n) \downarrow [s']$ with $s' < s$.

For each H_n, always start by putting $v_0 = (n, 0) \in H_n$. Let $0 \leq i < k$; we now define the $(i+1)^{\text{st}}$ vertex v_{i+1} of H_n. Assume that $\Delta^A_i(n) \downarrow [s]$ for the first time; recall that the range of $\Delta^A_i(n)$ is $\{0, \dots, r-1\}$. If it happens that $\Delta^A_0(n)$, $\Delta^A_1(n)$, $\dots$, $\Delta^A_i(n)$ constitute a valid coloring of the vertices $v_0, \dots, v_i$, then connect $v_{i+1} = (n, s+1)$ to the other vertices $(n, 0) = v_0$, v_1, $\dots$, v_i in H_n by using Alice's winning strategy against Bob's playing $\Delta^A_0(n), \dots, \Delta^A_i(n)$ in the first i rounds. If it does not constitute a valid coloring, then do not connect $v_{i+1} = (n, s+1)$ to any other vertices, and declare that $v_{i+1} \notin H_n$. Our full graph G will be computable, and so it will exist in our model.

Suppose for a contradiction that $\chi : \mathbb{N} \times \mathbb{N} \to r$ is a valid r-coloring of G. Define $\langle f_i \rangle_{i<k}$ as in the statement of $\mathsf{Predict}_k(r)$ by $f_i(n) = \chi(n, v_i)$ for $v_i \in H_n$. Notice that the f_i's are Σ^0_1-functions relative to A and χ, and that $\operatorname{dom} f_{i+1} = U_{i+1} = \operatorname{dom} \Delta^A_i$, for $v_{i+1} \in H_n$ is defined until $\Delta^A_i(n)$ refuses to halt. By $\mathsf{Predict}_k(r)$, there exists a particular n such that $f_i(n) = \Delta^A_i(n)$ for all $i < k$. This means in particular that there are a total of k vertices in H_n, namely $v_0, \dots, v_{k-1}$, and that χ constitutes a valid coloring on H_n. This means that if a_i encodes v_i and the edge relation with the previous vertices, we have $\langle a_0, \chi(v_0), a_1, \chi(v_1), \dots, a_{k-1}, \chi(v_{k-1}) \rangle$ is an outcome of the game $\mathsf{G}(\mathcal{C}, r)$ in k rounds that uses Alice's winning strategy, but is not winning for Alice, which is a contradiction.

Therefore, no such valid coloring χ exists, and G is a graph in $\mathcal{C}^{(k)}$ that is not r-colorable.

$\square$

The final thing to notice about Schmerl's construction is that it is a sequential graph construction in disguise. An infinite graph whose components have size at most k is essentially a sequence of finite graphs with a uniform bound k on the number of vertices. In fact, it takes a very minimal amount of proof modification to prove:

Theorem 2.4.8 (modified from Schmerl). *Let $r < \omega$. Let $\mathcal{C}$ be a natural class of graphs with a primitive recursive definition, and suppose that $\mathcal{C}$ is not on-line r-colorable. Then there exists $k < \omega$ such that if* $\mathsf{RCA_0} + \neg\mathsf{WKL_0}$ *holds, there exists a sequence of finite graphs $\langle G_n \rangle_{n \in \mathbb{N}}$, $G_n \in \mathcal{C}^{(k)}$, such that there does not exist a sequence $\langle \chi_n \rangle_{n \in \mathbb{N}}$ with χ_n a valid coloring $\chi_n : V_n \to r$.*

2.4.1 Schmerl's Example: A Non-2-Colorable Forest

In Section 1.2, we saw that the class of forests is not on-line 2-colorable, and in fact is not on-line r-colorable for any $r < \omega$. Of course, any forest is 2-colorable, and by Theorem 2.4.2, $\mathsf{WKL_0}$ is sufficient to prove this, since every forest is locally 2-colorable. But Theorem 2.4.4 shows that $\mathsf{WKL_0}$ is also necessary:

Corollary 2.4.9 (Schmerl). *Let $r < \omega$. Then the following is provable from* $\mathsf{RCA_0} + \neg\mathsf{WKL_0}$*: There is a forest that is not r-colorable.*

We also saw another class in Section 1.2 that is not on-line 2-colorable: the class of bipartite graphs that avoid the path P_6 of length 6. Bipartite graphs are locally 2-colorable, of course, and so $\mathsf{WKL_0}$ will suffice to prove that they are also 2-colorable. Once again the assumption of $\mathsf{WKL_0}$ was actually necessary:

Corollary 2.4.10. *Let $r < \omega$. Then the following is provable from* $\mathsf{RCA_0} + \neg\mathsf{WKL_0}$*: There is a locally 2-colorable graph in* $\mathrm{Forb}(P_6)$ *that is not r-colorable.*

Chapter 3

Characterizing the Reverse-Mathematical Strength of Sequential Problems

In this central chapter, we completely characterize the reverse-mathematical strength of a finite sequential problem.

In Section 2.4 we saw that if a class $\mathcal{C}$ of graphs is not on-line r-colorable, then WKL_0 is necessary to show that every infinite graph in $\mathcal{C}$ is r-colorable. In this chapter we generalize this, and show that the dividing line between RCA_0 and WKL_0 for a generic sequential problem is indeed on-line solvability.

Our first duty is to find a definition scheme that is capable of expressing all finite sequential problems in terms of a 2-player game. That is done in Section 3.1 below. All sequential problems can be viewed as a two-player game, even those cases (e.g. pigeonhole) that cannot possibly be on-line solvable. Section 3.2 is where we show that all bounded problems that are on-line solvable have sequential versions provable in RCA_0, whereas all standard-length problems that are not on-line solvable require

WKL_0 or something stronger.

Separating WKL_0 from ACA_0 requires a different set of concepts for its dividing line. We say that a problem has a *solvable closed kernel* if it has a solution, all of whose initial segments are solutions to the truncated problem. (The precise definition of this is in Section 3.1.) We will show that in a semi-bounded problem with a solvable closed kernel, WKL_0 suffices to prove the sequential version. If the closed kernel is not solvable, that is, if there is a length-k instance all of whose solutions fail at some initial segment, then ACA_0 is necessary to prove the sequential version if k is standard. If k is nonstandard, we will show that ACA_0 is necessary if we also assume $\mathsf{I}\Sigma^0_2$.

This separation is proved in Section 3.3. To prove it, we refine another technique of Schmerl's, using a device that we call a good-for-uniform k-tuple, whose existence is equivalent to $\neg\mathsf{ACA}_0$ for standard k. Schmerl introduced a similar concept in [34], but extending his construction to nonstandard k appears to require $\mathsf{I}\Pi^1_1$. In Subsection 3.3.5, we will show that Schmerl's construction at the very least require $\mathsf{I}\Sigma^0_2$. Schmerl applied this concept to prove results about Grundy colorings of graphs. In Section 3.4, we slightly improve his results.

3.1 Definitions related to sequential problems

By a *tree* we always mean a subtree of $\mathbb{N}^{<\mathbb{N}}$. If A and B are trees then $A \otimes B$ denotes the set of all pairs $(\bar{a}, \bar{b}) \in A \times B$ such that $lh(\bar{a}) = lh(\bar{b})$.

Definition 3.1.1. A *problem* is a triple (A, B, R) where A and B are trees and $R \subseteq A \otimes B$.

We will be thinking of elements of A as sequences $\bar{a} = \langle a_0, \ldots, a_{k-1}\rangle$ of questions by Alice, elements of B as sequences $\bar{b} = \langle b_0, \ldots, b_{k-1}\rangle$ of responses by Bob, and the relation $\bar{a}\ R\ \bar{b}$ holds if $\bar{b}$ is a list of correct answers to the questions $\bar{a}$.

For example, the problem of coloring a graph can be formalized in this way, with $\bar{a}$ representing a sequence that codes the vertices and edge relations, $\bar{b}$ representing a sequence of colors for the vertices, and $\bar{a}\ R\ \bar{b}$ holding if and only if the sequences define a proper graph coloring.

Definition 3.1.2. Given a problem (A, B, R) the game $\mathsf{G}(A, B, R)$ is played as follows: Alice and Bob alternate playing elements of $\mathbb{N}$:

$$\begin{array}{r|llllllll} \text{Alice} & a_0 & & a_1 & & a_2 & & \cdots \\ \hline \text{Bob} & & b_0 & & b_1 & & b_2 & \cdots \end{array}$$

Alice can stop the game at any time but Bob is required to respond to every one of Alice's plays. If the game stops after k rounds, then Bob wins if either $\langle a_0, \dots, a_{k-1}\rangle \notin A$ or $\langle a_0, \dots, a_{k-1}\rangle\ R\ \langle b_0, \dots, b_{k-1}\rangle$ holds; otherwise Alice wins. (So Bob wins if Alice never stops the game.)

In terms of strategies, Alice should (but is not required to) halt the game as soon as she reaches a point where $\langle a_0, \dots, a_{k-1}\rangle\ R\ \langle b_0, \dots, b_{k-1}\rangle$ fails. Similarly, Bob should (but is not required to) ensure that every partial play is such that $\langle a_0, \dots, a_{k-1}\rangle\ R\ \langle b_0, \dots, b_{k-1}\rangle$ holds unless $\langle a_0, \dots, a_{k-1}\rangle \notin A$.

Thus, the existence of a winning strategy for either player in $\mathsf{G}(A, B, R)$ is the same as the existence of a winning strategy for the same player in the game's closed kernel:

Definition 3.1.3. The *closed kernel* of a problem (A, B, R) is the problem (A, B, R'), where $\langle a_0, \dots, a_{k-1}\rangle\ R'\ \langle b_0, \dots, b_{k-1}\rangle$ holds when $\langle a_0, \dots, a_i\rangle\ R\ \langle b_0, \dots, b_i\rangle$ holds for every $i \leq k-1$. We can also talk about the closed kernel of the game $\mathsf{G}(A, B, R)$, which is the game $\mathsf{G}(A, B, R')$.

Definition 3.1.4. Given a problem (A, B, R), the statement $\mathsf{P}(A, B, R)$ is

$$\forall X(\alpha(X) \to \exists Y \beta(X, Y))$$

where:

- $\alpha(X)$ holds if X is a finite set of the form

$$\{(0, s_0, a_0),\ \ldots,\ (k-1, s_{k-1}, a_{k-1})\}$$

  where $s_0 < \cdots < s_{k-1}$ and $\langle a_0, \ldots, a_{k-1} \rangle \in A$, and

- $\beta(X, Y)$ holds if Y is a finite set of the form

$$\{(0, t_0, b_0),\ \ldots,\ (k-1, t_{k-1}, b_{k-1})\}$$

  where $t_0 < \cdots < t_{k-1}$ and $\langle a_0, \ldots, a_{k-1} \rangle \; R \; \langle b_0, \ldots, b_{k-1} \rangle$ holds.

We will imagine that $(k, s, a) \in X$ means that Alice asks a at time s and $(k, t, b) \in Y$ means that Bob responds b at time t. The first coordinate k indicates that there are exactly k earlier requests in the conversation, i.e., requests (k', s', a') where $k' < k$ and $s' < s$.

There is no requirement that this is a conversation: $s_0 \le t_0 < s_1 \le \cdots$ However, neither Alice nor Bob has any advantage in deviating from that natural sequence of events. Indeed, $\mathsf{P}(A, B, R)$ is equivalent to the stricter statement $\mathsf{P}'(A, B, R)$ where it is additionally required that $s_0 = t_0, \ldots, s_{k-1} = t_{k-1}$.

Definition 3.1.5. $\mathsf{SeqP}(A, B, R)$ is the statement

$$\forall X\, (\forall n\, \alpha(X_n) \;\to\; \exists Y\, \forall n\, \beta(X_n, Y_n))$$

where $\alpha(X)$ and $\beta(X, Y)$ are as in Definition 3.1.4. As usual, this notation means that $X = \langle X_n \rangle_{n\in\mathbb{N}}$ and $Y = \langle Y_n \rangle_{n\in\mathbb{N}}$.

Notice that we can uniformly enumerate Alice's sequences of requests X_n, but we cannot computably determine how long these sequences are.

Definition 3.1.6. $\mathsf{G}_k(A, B, R)$ is the game $\mathsf{G}(A \cap \mathbb{N}^{<k}, B \cap \mathbb{N}^{<k}, R \cap (\mathbb{N}^{<k} \otimes \mathbb{N}^{<k}))$.

Definition 3.1.7. $\mathsf{P}_k(A, B, R)$ is the statement $\mathsf{P}(A \cap \mathbb{N}^{<k}, B \cap \mathbb{N}^{<k}, R \cap (\mathbb{N}^{<k} \otimes \mathbb{N}^{<k}))$.

Definition 3.1.8. $\mathsf{SeqP}_k(A, B, R)$ is the statement $\mathsf{SeqP}(A \cap \mathbb{N}^{<k}, B \cap \mathbb{N}^{<k}, R \cap (\mathbb{N}^{<k} \otimes \mathbb{N}^{<k}))$.

Definition 3.1.9. A problem (A, B, R) is *solvable* if for every $\bar{a} \in A$ there is a $\bar{b} \in B$ such that $\bar{a} \; R \; \bar{b}$ holds. We say that (A, B, R) is *k-solvable* if every $\bar{a} \in A$ with length at most k there is a $\bar{b} \in B$ such that $\bar{a} \; R \; \bar{b}$ holds.

Definition 3.1.10. A problem (A, B, R) is *on-line solvable* if Bob has a winning strategy in $\mathsf{G}(A, B, R)$. We say that (A, B, R) is *on-line k-solvable* if Bob has a winning strategy in the restricted game $\mathsf{G}_k(A, B, R)$ where Alice is required to stop after the k^{th} round (or earlier).

Let us talk about what it means for the closed kernel (A, B, R') to be solvable. There are two equivalent interpretations that illustrate this idea. The first interpretation is that for every play $\bar{a}$ of Alice's, Bob has a winning play $\bar{b}$ such that every initial segment of that play is also winning. This is different from Bob having a winning strategy in the game, since Alice makes her full play immediately, with no input from Bob. Once again, the graph coloring problem is a useful example. If Alice presents a forest for Bob to 2-color, he can certainly color it such that all initial segments of his 2-coloring are valid, even though Bob does not have a winning strategy in the game, as we saw in Proposition 1.2.5.

The second interpretation is that Bob has a winning strategy in a game that creates an "on-line tree." Every time Alice plays, Bob adds one level to his tree of valid plays. A node of the tree is a dead end if Alice's next play makes it impossible for Bob to extend that node with any play. Bob wins if, once Alice has played all k of her plays, Bob has one or more branches of length k on his tree; that is, one or more winning plays. This winning play satisfies the first interpretation as well.

This second interpretation helps us visualize three cases of Bob's ability to solve a problem. Case 1 is that Bob has a uniform winning strategy; Case 2 is that Bob has a *tree* of winning plays as described above; Case 3 is that Bob has neither luxury, just an eventually winning play that will lose at some earlier round, for each corresponding play of Alice's. These three cases will correspond to the provability of the sequential problem in $\mathsf{RCA_0}$, $\mathsf{WKL_0}$, and $\mathsf{ACA_0}$, respectively.

Definition 3.1.11. The problem (A, B, R) is *semi-bounded* if Bob's valid responses are bounded by a function of Alice's previous plays. More precisely, there is a function f such that if $\langle a_0, \ldots, a_{k-1}\rangle \ R \ \langle b_0, \ldots, b_{k-1}\rangle$ holds then $b_0 < f\langle a_0\rangle$, $b_1 < f\langle a_0,\, a_1\rangle$, $\ldots$, $b_{k-1} < f\langle a_0,\, a_1,\, \ldots,\, a_{k-1}\rangle$.

Note that this is a requirement on R and not on B.

Definition 3.1.12. The problem (A, B, R) is *bounded* if, in addition to being semi-bounded, there is a function g such that for each $\bar{a} \in A$ and for each i we have $a_i < g(i)$. In other words, Alice's valid plays are bounded by g.

Note that this is a requirement on A.

3.2 Separating RCA_0 from WKL_0: On-Line Solvability

Proposition 3.2.1 (RCA_0)**.** *Let (A, B, R) be a problem which is not k-solvable. Then $\mathsf{SeqP}_k(A, B, R)$ fails.*

Proof. Let $\bar{a} \in A \cap \mathbb{N}^{<k}$ be a request such that there is no corresponding response $\bar{b} \in B \cap \mathbb{N}^{<k}$ with $\bar{a}\ R\ \bar{b}$. Consider the infinite constant sequence $\langle X_{\bar{a}}, X_{\bar{a}}, \ldots \rangle$, where $X_{\bar{a}}$ encodes the play $\bar{a}$. Then clearly there cannot exist $Y = \langle Y_n \rangle_{n \in \mathbb{N}}$ such that $\beta(X_n, Y_n)$ holds for even a single n. So $\mathsf{SeqP}_k(A, B, R)$ fails.

□

Theorem 3.2.2 (RCA_0)**.** *Let (A, B, R) be a problem which is on-line solvable. Then $\mathsf{SeqP}(A, B, R)$ holds.*

Proof. By assumption, Bob has a winning strategy. This gives a clear uniformly computable procedure to find the appropriate $\bar{b}$ which is winning against a given play $\bar{a}$. We can then easily produce a sequence $\langle Y_n \rangle_{n \in \mathbb{N}}$ of plays for Bob when given a sequence $\langle X_n \rangle_{n \in \mathbb{N}}$ of plays for Alice using this procedure, and we will have $\forall n\, \beta(X_n, Y_n)$. □

Definition 3.2.3. Let (A, B, R) be a bounded problem. Define the *maximum play* in the game $\mathsf{G}_k(A, B, R)$, denoted $M_k(A, B, R)$, as one more than the largest possible play by either Alice or Bob in the game $\mathsf{G}_k(A, B, R)$.

$M_k(A, B, R)$ is the maximum value of $g(i)$ and $f\langle a_0, \ldots, a_i \rangle$ as i ranges from 0 to k and such that for each j we have $a_j < g(j)$.

Lemma 3.2.4 (RCA_0)**.** *Let $k \geq 1$ and let (A, B, R) be a bounded problem. Then $\mathsf{G}_k(A, B, R)$ is determined.*

Proof. Let f, g be the functions witnessing the boundedness of the problem. Then in fact the full tree of Bob's strategies is bounded and computable, since both players' plays are bounded by $M_k(A, B, R)$.

So we can perform a search for a winning strategy for Bob through the whole tree, and we will have an answer in finite time. If we do not come up with a winning strategy, then Alice has a winning strategy. Namely, play a_0 such that Bob does not have a winning strategy above a_0, and in general play a_i such that Bob does not have a winning strategy above a_i. We can computably find this sequence $\bar{a}$ of plays, and $\bar{a}$ will hence be a winning play for Alice.

□

The theorem below generalizes the argument in Schmerl [33] for the graph coloring problem described in Section 2.4.

Theorem 3.2.5 (RCA_0)**.** *Let $k \in \mathbb{N}$. Let (A, B, R) be a bounded problem which is not on-line k-solvable. Let $M = M_k(A, B, R)$. If $\mathsf{Predict}_k(M + 1)$ holds, then $\mathsf{SeqP}_k(A, B, R)$ fails.*

Proof. Suppose that both $\mathsf{Predict}_k(M + 1)$ and $\mathsf{SeqP}_k(A, B, R)$ hold. Let W be an oracle such that $\Delta_0^W, \ldots, \Delta_{k-1}^W$ witness $\mathsf{Predict}_k(M + 1)$. We know by Lemma 3.2.4 that Alice has a winning strategy in $\mathsf{G}_k(A, B, R)$.

We will construct a sequence of instances of the finite game $\mathsf{G}_k(A, B, R)$: Alice will play a_i according to her winning strategy against Bob's playing the predictor sequence $\langle \Delta_j^W \rangle_{j<i}$ in every round $j < i$, as long as those potential plays are valid. In game n, Alice's plays are given by some X_n, and so by $\mathsf{SeqP}_k(A, B, R)$, there is a sequence $Y = \langle Y_n \rangle_{n \in \mathbb{N}}$ such that Y_n gives plays by Bob that win against Alice's plays in game n. Hence Alice's plays in game n win against the predictor sequence, since she used her winning strategy, but they lose against Y_n. However, the predictor sequence correctly predicts Y_n for some n, which is a contradiction.

Define $\langle X_n \rangle_{n\in\mathbb{N}}$ as follows: First let $(0, 0, a_0) \in X_n$, where a_0 is the initial play in Alice's winning strategy. For $0 < i < k$, assume that s is the first stage such that $\Delta^W_{i-1}(n)\downarrow [s]$. If for all j, $0 < j \le i$, by letting Bob play $\bar{b} = \langle \Delta^W_0(n),\ \Delta^W_1(n),\ \ldots,\ \Delta^W_{j-1}(n)\rangle$ against $\bar{a} = \langle a_0, a_1, \ldots, a_{j-1}\rangle$, where Alice plays a_j using her winning strategy at every step, we have $(\bar{a}\restriction j)\ R\ \bar{b}$ holding, then let a_i be Alice's next play in her winning strategy and add (i, s, a_i) to X_n. If there is any $j \le i$ such that the relation fails, we do not add any element to X_n. If $\Delta^W_{i-1}(n)\uparrow$, we do not add any element to X_n.

Given this sequence $\langle X_n \rangle_{n\in\mathbb{N}}$ of problem instances, let $\langle Y_n \rangle_{n\in\mathbb{N}}$ be the sequence of solutions guaranteed by $\mathsf{SeqP}_k(A, B, R)$. Define $\langle f_j \rangle_{j<k}$ according to two cases:

Case 1: $f_j(n) = b_j$, if $(j, t_j, b_j) \in Y_n$.

Case 2: $f_j(n) = M$ if there was $j' \le j$ such that $(\bar{a}\restriction j')\ R\ \langle \Delta^W_0(n), \ldots, \Delta^W_{j'-1}(n)\rangle$ fails, and we also have $\Delta^W_{j-1}(n)\downarrow$.

Case 1 and Case 2 are mutually exclusive. $\beta(X_n, Y_n)$ can only hold if X_n and Y_n have the same number of elements, and we would not add (j, s_j, a_j) to X_n if the condition of Case 2 held for some $j' \le j$.

Note that $\operatorname{dom} f_j = U_j = \operatorname{dom} \Delta^W_{j-1}$, since $f_j(n)$ will halt until the first time $\Delta^W_{j-1}(n)\uparrow$. Also recall that M is not a valid play for Alice or Bob, being one more that the maximum possible value in the game $\mathsf{G}_k(A, B, R)$.

By $\mathsf{Predict}_k(M+1)$, there is n such that $(\forall j < k)\, f_j(n) = \Delta^W_j(n)$. We have two cases.

Case 1': $f_j(n) = M$ for some $j < k$. This means that there is a least $j' \le j$ such that $(\bar{a}\restriction j')\ R\ \langle \Delta^W_0(n), \ldots, \Delta^W_{j'-1}(n)\rangle$ fails. We also have $\Delta^W_i(n) = f_i(n)$ for $0 \le i \le j'-1$. By the minimality of j', we are in Case 1 above for f_i, $0 \le i \le j'-1$; in other words, Bob plays j' times and $\bar{b} = \langle b_0, \ldots, b_{j'-1}\rangle = \langle f_0(n), \ldots, f_{j'-1}(n)\rangle =$

$\langle \Delta_0^W(n), \ldots, \Delta_{j'-1}^W(n) \rangle$, and therefore $(\bar{a} \upharpoonright j')\ R\ \bar{b}$ fails. But in this case $(\bar{a} \upharpoonright j')$ is Alice's entire play in X_n, and so $\beta(X_n, Y_n)$ fails, contradicting that $\beta(X_n, Y_n)$ holds.

Case 2': $f_j(n) < M$ for all $j < k$, which means that there are a full k elements in Y_n. This means that Bob plays k times, and $\bar{b} = \langle \Delta_0^W(n), \ldots, \Delta_{k-1}^W(n) \rangle$ is such that $(\bar{a} \upharpoonright k)\ R\ \bar{b}$ holds; that is, $\bar{a}\ R\ \bar{b}$ holds. However, this means that $\bar{b}$ is a winning play against Alice's winning strategy, which is a contradiction.

□

Corollary 3.2.6 ($\mathsf{RCA_0}$). *Let $k < \omega$, and let (A, B, R) be a bounded problem which is not on-line k-solvable. Also assume that $M < \omega$, where $M = M_k(A, B, R)$. Then $\mathsf{SeqP}_k(A, B, R)$ implies $\mathsf{WKL_0}$.*

Proof. Corollary 2.3.13 and Theorem 3.2.5.

□

3.2.1 Determinacy of the relevant games

In proving Theorem 3.2.5, we needed the fact that the finite game $\mathsf{G}_k(A, B, R)$ is determined when (A, B, R) is bounded. In this subsection we present some further determinacy results about the related games.

Lemma 3.2.7 ($\mathsf{WKL_0}$). *Let (A, B, R) be a bounded problem. Then if Bob has a winning strategy in every game $\mathsf{G}_k(A, B, R)$ for $k \in \mathbb{N}$, then Bob has a winning strategy in $\mathsf{G}(A, B, R)$.*

Proof. We will define a tree T of winning strategies for Bob. Each node of T on level j will encode Bob's response to *every possible* sequence of plays by Alice. Let f, g witness the boundedness of Bob's and Alice's plays as presented in Definitions 3.1.11 and 3.1.12.

If $\sigma \in T$, then $\sigma(j)$ is a sequence $\langle b_{j,0},\, b_{j,1},\, \ldots,\, b_{j,N-1}\rangle$, such that N is the number of possible sequences of Alice's plays in rounds 0 through j, and $b_{j,i} \in B$ is Bob's valid response (meaning the relation R holds) to Alice's i^{th} sequence of plays in some canonical ordering. This is, of course, assuming that when Alice played the first $j' < j$ plays in her i^{th} sequence, Bob played the corresponding $b_{j',i'}$ encoded in $\sigma(j')$, which is $\sigma(j)$'s predecessor on level j' of the tree.

If for even one i, $0 \leq i < N$, Bob does not have a winning response, then the tree has a dead end, and $\sigma(j)$ is undefined.

By assumption, Bob has a winning strategy in $\mathsf{G}_j(A, B, R)$ for every j. This means that T is infinite, and an infinite branch of T would clearly encode a full winning strategy for Bob in $\mathsf{G}(A, B, R)$.

The tree is also bounded. Suppose that $\sigma(j)$ is a sequence of length N and its code has value M. I claim that in the recursive function $h(j, r)$ defined below, $h(j, 0)$ is a bound for N (number of possible plays for Alice in the first j steps) and $h(j, 1)$ is a bound for M (coded value of the sequence of possible responses by Bob in round j):

$$
\begin{aligned}
h(0,0) &= g(0) \\
h(0,1) &= f(g(0))^{h(0,0)} \\
h(1,0) &= h(0,0) \cdot g(1) \\
h(1,1) &= f(g(0),\, g(1))^{h(1,0)} \\
&\cdots \\
h(j,0) &= h(j-1,0) \cdot g(j) \\
h(j,1) &= f(g(0),\, g(1),\, \ldots,\, g(j))^{h(j,0)}
\end{aligned}
$$

WKL_0 shows that this tree has an infinite path, and this path will encode a winning strategy for Bob in $\mathsf{G}(A, B, R)$.

□

Theorem 3.2.8 ($\mathsf{WKL_0}$)**.** *Let (A, B, R) be a bounded problem. Then $\mathsf{G}(A, B, R)$ is determined.*

Proof. If Bob does not have a winning strategy in $\mathsf{G}(A, B, R)$, then Lemma 3.2.7 implies that there exists $k \in \mathbb{N}$ such that Bob does not have a winning strategy in $\mathsf{G}_k(A, B, R)$. Then Lemma 3.2.4 implies that Alice has a winning strategy in $\mathsf{G}_k(A, B, R)$ and hence in $\mathsf{G}(A, B, R)$.

□

Lemma 3.2.9 ($\mathsf{WKL_0}$)**.** *Let (A, B, R) be a semi-bounded problem. Then Alice has a winning strategy in $\mathsf{G}(A, B, R)$ if and only if there exist $c, k \in \mathbb{N}$ such that Alice has a winning strategy in $\mathsf{G}_k(A \cap c^{<k}, B, R)$, where all of Alice's plays are required to be in the set $\{0, \ldots, c-1\}$.*

Proof. Suppose that Alice has a winning strategy in $\mathsf{G}(A, B, R)$.

I claim that plays in this winning strategy have a maximum possible length k. Define a tree T of plays of the game in which Alice always uses her winning strategy. If $\sigma \in T$, then $\sigma(2j)$ encodes a play by Alice using her winning strategy, given earlier plays $\sigma(0), \ldots, \sigma(2j-1)$ by both players. (This includes the root $\sigma(0)$, which is an initial play in Alice's winning strategy). $\sigma(2j+1)$ encodes any play by Bob given earlier plays $\sigma(0), \ldots, \sigma(2j)$ by both players. In this tree, a node $\sigma(2j)$ is a dead end if Bob does *not* have a valid response $\sigma(2j+1)$ to the earlier plays $\langle\sigma(0), \sigma(1), \ldots, \sigma(2j)\rangle$.

Now the problem is only semi-bounded, so we have a bound f for Bob's plays as presented in Definition 3.1.11, but no bound for Alice's plays. However, I claim that for $\sigma \in T$, $\sigma(j)$ is bounded by $h(j)$, where $h(j)$ is the recursive function defined below:

$$
\begin{aligned}
h(0) &= \text{Alice's initial play in her winning strategy} \\
h(1) &= f(h(0)) \\
h(2) &= \text{maximum response in Alice's winning strategy to} \\
& \text{Bob playing } 0, \ldots, h(1) \\
&\cdots \\
h(2j+1) &= \max\left\{ f(a_0, a_1, \ldots, a_j) : 0 \le a_i \le h(2i) \right\} \\
h(2j+2) &= \text{maximum response in Alice's winning strategy to} \\
& \text{Bob playing } 0, \ldots, h(2i+1) \text{ in the } i^{\text{th}} \text{ round}, 0 \le i \le j
\end{aligned}
$$

So the levels of T are bounded. However, in this case T is not infinite, for if we had an infinite branch of T, then Bob has an infinite sequence of valid responses, and therefore Bob can win the game, contradicting that Alice has a winning strategy. Therefore, T has a maximum possible level k.

Therefore, there are only a finite number of possible plays by Bob if we assume Alice repeatedly uses her strategy. In fact, we can use $h(j)$ above to put a bound on values of Alice's plays just as we can in a fully bounded problem: $a_j < h(2j)$. If we choose $c = h(2k)$, then Alice has a winning strategy in $\mathsf{G}_k(A \cap c^{<k}, B, R)$ as desired.

□

Theorem 3.2.10 ($\mathsf{WKL_0}$)**.** *Let (A, B, R) be a semi-bounded problem. Then $\mathsf{G}(A, B, R)$ is determined.*

Proof. If Alice does not have a winning strategy, then Lemma 3.2.9 implies that for all $c, k \in \mathbb{N}$, Alice does not have a winning strategy in $\mathsf{G}_k(A \cap c^{<k}, B, R)$. But $(A \cap c^{<k}, B \cap \mathbb{N}^{<k}, R \cap (\mathbb{N}^{<k} \otimes \mathbb{N}^{<k}))$ is a bounded problem, and so Lemma 3.2.4 implies that Bob does have a winning strategy in $\mathsf{G}_k(A \cap c^{<k}, B, R)$ for *every* c, k.

We can essentially follow the proof of Lemma 3.2.7, gluing Bob's finite strategies together to form a tree, to show that he has a winning strategy in $\mathsf{G}(A, B, R)$. To show that the tree is infinite, just note that for every $k \in \mathbb{N}$ there is a winning play for Bob in $\mathsf{G}_k(A, B, R)$, since if $c = g(k)$ is as we constructed in Lemma 3.2.9, then Bob's winning strategy in $\mathsf{G}_k(A \cap c^{<k}, B, R)$ is necessarily also a winning strategy in $\mathsf{G}_k(A, B, R)$.

$\square$

Remark 3.2.11. It is *not* the case that if (A, B, R) is a solvable problem, then $\mathsf{ACA_0}$ proves that $\mathsf{G}(A, B, R)$ is determined. This statement requires $\mathsf{ATR_0}$.

3.3 Separating $\mathsf{WKL_0}$ from $\mathsf{ACA_0}$: Solvability of the closed kernel

The characteristic that separates the sequential problems provable in $\mathsf{WKL_0}$ from those that require $\mathsf{ACA_0}$ is the solvability of the closed kernel.

First let us note that $\mathsf{ACA_0}$ suffices to prove $\mathsf{SeqP}(A, B, R)$ for every solvable problem (A, B, R). Recall that D_n stands for the n^{th} finite set in a canonical ordering.

Proposition 3.3.1 ($\mathsf{ACA_0}$)**.** *Let (A, B, R) be a solvable problem. Then $\mathsf{SeqP}(A, B, R)$ holds.*

Proof. Let $\langle X_n \rangle_{n\in\mathbb{N}}$ be a sequence such that $\forall n\, \alpha(X_n)$. Since each X_n encodes a series of requests $\bar{a}$ for Alice in a solvable problem, we know that for each n there exists $\bar{b}$ such that $\bar{a}\ R\ \bar{b}$, so we can encode a Y_n such that $\beta(X_n, Y_n)$.

We can use $\mathsf{ACA_0}$ to define a function which gives an upper bound $b(n)$ for each finite set X_n, and then using the upper bounds, we can find a function f that codes each finite set X_n as $D_{f(n)}$. ($D_{f(n)}$ can be found by considering all finite subsets of $\{0, \ldots, b(n)\}$.) Then we can use minimization to find the least e_n such that $\beta(D_{f(n)}, D_{e_n})$ holds. Define $Y := \langle Y_n \rangle_{n \in \mathbb{N}} = \langle D_{e_n} \rangle_{n \in \mathbb{N}}$ and we are done.

□

3.3.1 Cases where WKL_0 is sufficient

We show in the following theorem that for a semi-bounded problem, if the closed kernel is solvable, then WKL_0 is sufficient to prove the sequential problem.

Theorem 3.3.2 (WKL_0)**.** *Let (A, B, R) be a semi-bounded problem with closed kernel (A, B, R'). If (A, B, R') is solvable, then $\mathsf{SeqP}(A, B, R)$ holds.*

Proof. Let $\langle X_n \rangle_{n \in \mathbb{N}}$ be a sequence such that $\forall n\, \alpha(X_n)$.

We will enumerate triples $\langle n, s, a \rangle$ according to a fixed bijective pairing function $p : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, such that if $p(m) = (n, 0, 0)$ and $p(m') = (n', 0, 0)$ with $m < m'$, then $n < n'$.

Define a tree T as follows: $\sigma \in T$ if and only if for all $\langle n, s, a \rangle < lh(\sigma)$:

- If $(\exists k \leq s)\, (k, s, a) \in X_n$, then

$$\langle a_0, a_1, \ldots, a_k \rangle \ R \ \langle \sigma(n, s_0, a_0), \sigma(n, s_1, a_1), \ldots, \sigma(n, s_k, a_k) \rangle$$

  where $(k, s_k, a_k) = (k, s, a)$ and $(0, s_0, a_0), (1, s_1, a_1), \ldots, (k, s_{k-1}, a_k)$ is the complete list of elements of X_n with first coordinate $k \leq s$.

- If $(\forall k \leq s)(k, s, a) \notin X_n$, then $\sigma(n, s, a) = 0$.

This tree T is computable from the given sequence $\langle X_n \rangle_{n \in \mathbb{N}}$. Even though there is no bound on the a_i's, since we know that there are exactly k triples $(i, s_i, a_i) \in X_n$ with $i < k$ and $s_i < s$, we can do a computable search for the required a_i's.

We will now show that the tree is also infinite. Let $n \in \mathbb{N}$; we will show that T has a node of length $\langle n, 0, 0\rangle$. Consider all requests less than X_n: the requests X_0, X_1, …, X_{n-1}. Some of these requests may have been fully enumerated, and others may have only been partially enumerated, before $\langle n, 0, 0\rangle$. That is, for $n' < n$, the set $\{(k, s, a) : \langle n', s, a\rangle < \langle n, 0, 0\rangle\}$ may contain every $(k, s, a) \in X_{n'}$ or it may not. However, since the closed kernel is solvable, there exists a solution $\bar{b}$ to the full $\bar{a}$ encoded by $X_{n'}$ (i.e., $\bar{a}\ R\ \bar{b}$), such that every initial segment of $\bar{b}$ is also a solution to the corresponding initial seqment of $\bar{a}$ (i.e., $(\bar{a} \restriction j)\ R\ (\bar{b} \restriction j)$ for every j). For this reason, T indeed has a node of length $\langle n, 0, 0\rangle$, a node that encodes a valid partial or total solution to all plays $X_{n'}$, $n' < n$.

Finally, T is a bounded tree. By the definition above, if $\sigma \in T$, $\sigma(n, s, a)$ is either 0 or is of the form b_r such that $\langle a_0, \ldots, a_r\rangle\ R\ \langle b_0, \ldots, b_r\rangle$ for appropriate a_0, …, a_r, b_0, …, b_{r-1}. Since our problem is semi-bounded, we have
$b_r < f\langle a_0, a_1, \ldots, a_r\rangle$ for some function f in our model, and the a_i's come from our (given) sequence $\langle X_n\rangle_{n\in\mathbb{N}}$. So $\sigma(n, s, a)$ is bounded by a function in our model, and T is a bounded tree.

By $\mathsf{WKL_0}$, T has an infinite branch $h \in [T]$. Given X_n, we can define Y_n by: $(k, s, b) \in Y_n$ if and only if there exists a with $(k, s, a) \in X_n$ and $h(n, s, a) = b$. It is clear that $\beta(X_n, Y_n)$ holds. So $\langle Y_n\rangle_{n\in\mathbb{N}}$ is a solution to $\mathsf{SeqP}(A, B, R)$.

□

3.3.2 A special case where $\mathsf{ACA_0}$ is necessary

Conversely, it is precisely the problems without solvable closed kernels which require $\mathsf{ACA_0}$ to prove the sequential problem. Proving this will take a bit more work. There is a relatively easy proof in the following special case:

Proposition 3.3.3 (RCA_0)**.** *Let $k \in \mathbb{N}$. Suppose that $P = (A, B, R)$ is a solvable problem whose closed kernel (A, B, R') is not k-solvable. Suppose, in addition, that there is a request $\bar{a}$ of length k and a particular $j < k$ such that for each $\bar{b}$ with $\bar{a}\ R\ \bar{b}$, we have $(\bar{a} \restriction j)\ R\ (\bar{b} \restriction j)$ fails. Then if $\mathsf{SeqP}(A, B, R)$ holds, ACA_0 holds.*

Note that this includes all cases where each request $\bar{a}$ has a unique solution $\bar{b}$.

Proof. Let $\bar{a} = \langle a_0, \ldots, a_{k-1} \rangle$ and $j < k$ be as stated above. Let $f : \mathbb{N} \to \mathbb{N}$ be an arbitrary injection; we will show that $\operatorname{ran} f$ exists.

Define a sequence $\langle X_n \rangle_{n \in \mathbb{N}}$ as follows: $(0, 0, a_0)$, $(1, 1, a_1)$, $\ldots$, $(j-1, j-1, a_{j-1})$ are always in X_n. If $f(s) = n$, then $(j, s+j, a_j)$, $\ldots$, $(k-1, s+k-1, a_{k-1})$ are also in X_n. Note that $\langle X_n \rangle_{n \in \mathbb{N}}$ is a computable sequence and that $\alpha(X_n)$ holds for all n.

By $\mathsf{SeqP}_k(A, B, R)$, there exists $\langle Y_n \rangle_{n \in \mathbb{N}}$ such that $\forall n \beta(X_n, Y_n)$ holds. Given $n \in \mathbb{N}$, check whether $(\bar{a} \restriction j)\ R\ (\bar{b} \restriction j)$ holds. (Here $\bar{b} = \langle b_0, \ldots, b_r \rangle$ is such that $Y_n = \{(0, t_0, b_0), \ldots, (r, t_r, b_r)\}$ for some r, $j - 1 \le r \le k - 1$.) If this relation does hold, then by assumption $\bar{b} \restriction j$ cannot extend to a full solution $\bar{b}$ with $\bar{a}\ R\ \bar{b}$, and so $n \notin \operatorname{ran} f$. On the other hand, if the relation does not hold, then since Y_n is a winning response to X_n, $\bar{b} \restriction j$ must extend to a full solution $\bar{b}$ with $\bar{a}\ R\ \bar{b}$, and so $n \in \operatorname{ran} f$.

□

3.3.3 Good Tuples and Good-for-Uniform Tuples

Not all applications satisfy the requirements of Proposition 3.3.3; it is possible that each request $\bar{a}$ has multiple winning responses $\bar{b_1}$, $\bar{b_2}$ where initial segments of different lengths fail to belong to R. For examples of this, see the non-optimal pigeonhole principle (Theorem 4.1.2) and the task scheduling problem (Theorem 4.4.10) in the next chapter. Analyzing problems of this type is the goal of this subsection, and it requires two new concepts: the "good tuple" introduced by Schmerl, and our modification, the "good-for-uniform tuple."

The following Definition 3.3.5 is a slight modification of the one Schmerl uses in [34]. We will work in a model $\mathcal{N} = (\mathbb{N}, \mathfrak{X})$ of RCA_0.

Definition 3.3.4. We say that $X \subseteq \mathbb{N}$ is *enumerable* if either it is finite or there is an injective function in $\mathfrak{X}$, $e_X : \mathbb{N} \to \mathbb{N}$, which enumerates X:

$$x \in X \quad \leftrightarrow \quad \exists n \, (e_X(n) = x)$$

Recall that $\mathcal{N} \models \mathsf{ACA}_0$ if and only if every enumerable set is in $\mathfrak{X}$.

Definition 3.3.5. Let $n \geq 2$. The n-tuple $\langle X_0, X_1, \ldots, X_{n-1} \rangle$ is *good* if $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_{n-1}$, each X_i is enumerable, and whenever $Y_1, Y_2, \ldots, Y_{n-1}$ are enumerable sets such that, for $1 \leq i \leq n-1$, $Y_i \subseteq X_i \setminus (Y_1 \cup Y_2 \cup \cdots \cup Y_{i-1})$ and $X_{i-1} \setminus (Y_1 \cup Y_2 \cup \cdots \cup Y_i)$ is enumerable, then $X_{n-1} \setminus (Y_1 \cup Y_2 \cup \cdots \cup Y_{n-1}) \neq \varnothing$.

Schmerl's definition of "good" requires only $(n-2)$ of the Y_i's in the condition; in our definition both X_i and Y_i range through $i = n-1$. In our definition, there can be good 2-tuples; in Schmerl's defnition, the smallest size is a 3-tuple.

As Schmerl notes, if $\langle X_0, X_1, \ldots, X_{n-1} \rangle$ is a good tuple, then X_0 cannot be finite (otherwise just take $Y_1 = X_1$, $Y_i = \varnothing$ for $i > 1$), meaning that X_0 is the range of an injective function $e_{X_0} : \mathbb{N} \to \mathbb{N}$, and it is easy to check that
$\langle e_{X_0}^{-1}(X_0),\, e_{X_0}^{-1}(X_1),\, \ldots,\, e_{X_0}^{-1}(X_{n-1}) \rangle$ is also good. So if we are assuming the existence of a good k-tuple, we are free to choose $X_0 = \mathbb{N}$.

Lemma 3.3.6 (Schmerl [34], RCA_0)**.** *Let $2 \leq n < \omega$. Then $\mathcal{N} \models \mathsf{ACA}_0$ if and only if there are no good n-tuples.*

Extending Schmerl's argument to nonstandard n appears to require Π^1_1-induction, the culprit being the clause "whenever $Y_1, \ldots, Y_{n-1}$ are enumerable" in the hypothesis.

We introduce a slightly weaker concept called a "good-for-uniform" tuple, and this will allow us to prove an analogue of Lemma 3.3.6 with much milder induction: our proof will hold in RCA_0 for standard $n < \omega$ and in $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$ for nonstandard $n \in \mathbb{N}$.

Definition 3.3.7. Let $n \geq 2$. The n-tuple $\langle X_0, X_1, \dots, X_{n-1} \rangle$ is *good-for-uniform* if $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_{n-1}$, each X_i is enumerable, and whenever $\langle f_0, f_1, \dots, f_{n-2} \rangle$ is a sequence of partial $\mathbf{\Sigma}^0_1$-functions $f_i : X_i \to \{0, 1, \dots, i+1\}$ such that $f^{-1}_{i+1}\{j\} = f^{-1}_i\{j\} \cap X_{i+1}$ for all $j \leq i < n-2$ and $f^{-1}_i\{i\} \subseteq X_{i+1}$ for $i < n-1$, then we have $X_{n-1} \cap f^{-1}_{n-2}\{n-1\} \neq \varnothing$.

When we say that f_i is a partial $\mathbf{\Sigma}^0_1$-function $f_i : X_i \to \{0, \dots, i+1\}$, we mean that the domain is all of X_i, so it is not "partial" on X_i. We just emphasize that "$f_i(x) = j$" is a $\mathbf{\Sigma}^0_1$-statement, since X_i may not be a set in our model $\mathcal{N}$. Also, note that our sequence $\langle f_0, f_1, \dots, f_{n-2} \rangle$ can be viewed as a single $\mathbf{\Sigma}^0_1$-function $f : \mathbb{N} \times X_0 \to \{0, \dots, n-1\}$.

Proposition 3.3.8. *If $\langle X_0, X_1, \dots, X_{n-1} \rangle$ is good, then it is good-for-uniform.*

Proof. Let $\langle f_0, \dots, f_{n-2} \rangle$ be a sequence of functions with the properties in Definition 3.3.7.

If we define $Y_1, \dots, Y_{n-1}$ and $Z_0, \dots, Z_{n-2}$ as follows:

$$Y_1 = f^{-1}_0\{0\} \qquad Z_0 = f^{-1}_0\{1\}$$

$$Y_2 = f^{-1}_1\{1\} \qquad Z_1 = f^{-1}_1\{2\}$$

$$\dots$$

$$Y_{n-1} = f^{-1}_{n-2}\{n-2\} \qquad Z_{n-2} = f^{-1}_{n-2}\{n-1\}$$

then they satisfy the conditions in the definition of a good n-tuple: $Y_1, Y_2, \dots, Y_{n-1}$

are enumerable sets, $Y_i \subseteq X_i$, $Z_{i-1} = X_{i-1} \setminus (Y_1 \cup Y_2 \cup \cdots \cup Y_i)$ is also enumerable, and $Y_i \cap (Y_1 \cup Y_2 \cup \cdots \cup Y_{i-1}) = \varnothing$.

Since our tuple is good, we have $X_{n-1} \setminus (Y_1 \cup Y_2 \cup \cdots \cup Y_{n-1}) \neq \varnothing$, or equivalently, $X_{n-1} \cap f_{n-2}^{-1}\{n-1\} \neq \varnothing$.

□

A proof of the following lemma, for good tuples, is presented in Schmerl [34]. However, there appears to be a gap in this proof. The author would like to thank Groszek and Slaman [15] for outlining a correction to Schmerl's proof. The following proof incorporates their corrections and generalizes to good-for-uniform tuples.

Lemma 3.3.9 ($\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$)**.** *Fix an enumerable set A and $m \geq 2$. We will use a pairing function to identify $\mathbb{N}^m$ with $\mathbb{N}$. For $i < m$, let $X_i = A^i \times \mathbb{N}^{m-1-i}$ (so, in particular, $X_0 = \mathbb{N}^{m-1}$ and $X_{m-1} = A^{m-1}$). If $\langle X_0, X_1, \ldots, X_{m-1}\rangle$ is not a good-for-uniform m-tuple, then $\mathbb{N} \setminus A$ is enumerable.*

Proof. Suppose that $f = f(i, n)$ is the partial function that witnesses that $\langle X_0, X_1, \ldots, X_{m-1}\rangle$ is not good-for-uniform, so that:

- $f(i, n) = f_i(n)$, $0 \leq i \leq m-2$
- $f_i : X_i \to \{0, \ldots, i+1\}$
- $f_i(x) = j < i \ \to \ f_{i-1}(x) = j$
- $f_i(x) = i \ \to \ x \in X_{i+1}$
- $\operatorname{ran}(f_{m-2} \restriction X_{m-1}) \subseteq \{0, \ldots, m-2\}$

Let $j < m-2$. Define $\bar{w} = \left\langle w_{(m-2)-(j-1)}, w_{(m-2)-(j-2)}, \ldots, w_{m-2}\right\rangle$ to be a *bad sequence* if each w_i is in $\mathbb{N} \setminus A$ and for every sequence $\bar{v} = \left\langle v_0, \ldots, v_{(m-2)-j}\right\rangle$ either

$\bar{v} \notin A^{(m-1)-j}$ or

$$f_{(m-2)-j}(v_0, \dots, v_{(m-2)-j}, w_{(m-2)-(j-1)}, w_{(m-2)-(j-2)}, \dots, w_{m-2}) \neq (m-1)-j.$$

Since A is enumerable, the set of all bad sequences is a Π^0_1 set. Clearly the empty sequence is a bad sequence.

Case 1: There exists a bad sequence $\langle w_1, w_2, \dots, w_{m-2}\rangle$ of length $m-2$.

Let $v \in \mathbb{N}$. If $v \in A$, then $f_0(v, w_1, w_2, \dots, w_{m-2}) \neq 1$, meaning

$$f_0(v, w_1, w_2, \dots, v_{m-2}) = 0.$$

On the other hand, if $v \notin A$, then $(v, w_1, w_2, \dots, w_{m-2}) \notin \operatorname{dom} f_1$. This means that $f_0(x) \neq 0$, implying that

$$f_0(v, w_1, w_2, \dots, v_{m-2}) = 1.$$

Therefore, since $\operatorname{dom} f_0 = \mathbb{N}$, A is computable; hence $\mathbb{N} \setminus A$ is enumerable.

Case 2: There are no bad sequences of length $m-2$.

Let $j < m-2$ be the greatest j such that there exists a bad sequence of length j. Note that the statement $\Theta(j) :=$ "There exists $j' \geq j$ and a bad sequence $\bar{w}$ of length j'" is a Σ^0_2 statement, and by $\mathsf{I}\Sigma^0_2$, since $\forall j \Theta(j)$ fails and since $\Theta(0)$ holds, there exists j_0 such that $\Theta(j_0) \wedge \neg\Theta(j_0+1)$ holds.

We claim that $\mathbb{N} \setminus A$ is enumerable; namely, $v \in \mathbb{N} \setminus A$ if and only if

$$\left[\exists \left\langle a_0, \dots, a_{(m-3)-j}\right\rangle \in A^{(m-2)-j}\right]$$

$$f_{(m-2)-j}(a_0, \dots, a_{(m-3)-j}, v, w_{(m-2)-(j-1)}, \dots, w_{m-2}) = (m-1)-j$$

For if $v \in A$, then since $\bar{w}$ is bad, for any $\bar{a} \in A^{(m-2)-j}$, we have $f_{(m-2)-j}(\bar{a}, v, \bar{w}) \neq (m-1)-j$. On the other hand, if $v \in \mathbb{N} \setminus A$, then if for all $\bar{a} \in A^{(m-2)-j}$ we had $f_{(m-2)-j}(\bar{a}, v, \bar{w}) \neq (m-1)-j$, then $(v, \bar{w})$ would be a bad sequence of length $j+1$, a contradiction.

$\square$

Corollary 3.3.10 (RCA_0)**.** *If $m < \omega$, then Lemma 3.3.9 can be proven in RCA_0. That is, suppose that $m < \omega$, A is enumerable, and $X_i = A^i \times \mathbb{N}^{m-1-i}$ for $0 \leq i \leq m-1$. If $\langle X_0, X_1, \ldots, X_{m-1} \rangle$ is not a good-for-uniform m-tuple, then $\mathbb{N} \setminus A$ is enumerable.*

Proof. Observe that in the proof of Lemma 3.3.9, we divided our argument into 2 cases and applied Σ^0_2-induction in the second case. However, if $m < \omega$, then we can divide the argument into exactly $m-1$ cases:

Case 1: There exists a bad sequence of length $m-2$.

Case 2: There exists a bad sequence of length $m-3$ but none of higher length.

...

Case i: There exists a bad sequence of length $(m-1)-i$ but none of higher length.

...

Case $m-1$: The only bad sequence is the empty sequence of length 0.

Or, if you like, since $\Theta(0)$ is true, we know that the following sentence is true:

$$\bigvee_{i=0}^{m-2} \Theta(i) \wedge [(\forall i', i < i' < m-2) \neg\Theta(i')]$$

In each case, we can check that $\mathbb{N} \setminus A$ is enumerable using the exact method presented in the proof of the previous lemma.

$\square$

Theorem 3.3.11 ($\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$)**.** *Let $m \in \mathbb{N}$. Then ACA_0 holds if and only if there are no good-for-uniform m-tuples.*

Proof. Let $\mathcal{N} = (\mathbb{N}, \mathfrak{X})$ be a model of RCA_0. Suppose that $\mathcal{N} \models \mathsf{ACA}_0$ and suppose that $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_{m-1}$, with each X_i enumerable. To show that the m-tuple is not good-for-uniform, define the sequence $\langle f_0, \ldots, f_{m-2}\rangle$ by $f_i(X_1) = 0$ for all i; $f_0(X_0 \backslash X_1) = 1$. Since X_1 is arithmetical, ACA_0 proves the existence of the tuple $\langle f_0, \ldots, f_{m-2}\rangle$. Then $\langle X_0, X_1, \ldots, X_{m-1}\rangle$ satisfies the hypotheses of a good-for-uniform m-tuple but not its conclusion, since $X_{m-1} \subseteq X_1$ and thus $f_{m-2}(X_{m-1}) = 0$.

Now assume that $\mathcal{N} \models \neg\mathsf{ACA}_0$. Fix an enumerable set A such that $A \notin \mathfrak{X}$, and let $\langle X_0, X_1, \ldots, X_{m-1}\rangle$ be defined as in Lemma 3.3.9, so that $X_i = A^i \times \mathbb{N}^{m-1-i}$ for $i < m$. Since $\mathbb{N} \setminus A$ is not enumerable, then the lemma shows that $\langle X_0, X_1, \ldots, X_{m-1}\rangle$ is a good-for-uniform tuple.

□

Corollary 3.3.12 (RCA_0)**.** *Let $m < \omega$. Then ACA_0 holds if and only if there are no good-for-uniform m-tuples. In other words, for standard $m < \omega$, RCA_0 is sufficient to prove Theorem 3.3.11.*

Proof. This follows from the proof of Theorem 3.3.11, but with Lemma 3.3.9 replaced with Corollary 3.3.10.

□

3.3.4 The general case where ACA_0 is necessary

Theorem 3.3.13 (Dorais-Harris, RCA_0)**.** *Let $k \in \mathbb{N}$. Suppose we are given a problem (A, B, R) and its closed kernel (A, B, R'). If (A, B, R') is not k-solvable and there is a good-for-uniform k-tuple, then $\mathsf{SeqP}_k(A, B, R)$ fails.*

Proof. Assume that (A, B, R) is k-solvable, but (A, B, R') is not k-solvable. (If (A, B, R) is not k-solvable, then $\mathsf{SeqP}_k(A, B, R)$ fails automatically.) Let $\langle a_0, \ldots, a_{k-1} \rangle$ be a request from Alice such that for any winning response $\langle b_0, \ldots, b_{k-1} \rangle$ (meaning that $\bar{a}\ R\ \bar{b}$), there exists $j < k$ such that $\langle a_0, \ldots, a_{j-1} \rangle\ R\ \langle b_0, \ldots, b_{j-1} \rangle$ does not hold. Also let $\langle X_0, \ldots, X_{k-1} \rangle$ be a good-for-uniform k-tuple with $X_0 = \mathbb{N}$.

Now also assume that $\mathsf{SeqP}_k(A, B, R)$ holds. Define $\langle A_n \rangle_{n \in \mathbb{N}}$ as follows: $(i,\ i + s_0 + \cdots + s_i,\ a_i) \in A_n$ if and only if $e_{X_{i'}}(s_{i'}) = n$ for all $i' \leq i$, and a_i is in Alice's above-mentioned request. So the sequence of requests in A_n will be $\langle a_0, \ldots, a_i \rangle$ precisely if $n \in X_i \backslash X_{i+1}$. Note that $\langle A_n \rangle_{n \in \mathbb{N}}$ is computable and $\alpha(A_n)$ holds for all n.

By $\mathsf{SeqP}_k(A, B, R)$, there exists $\langle B_n \rangle_{n \in \mathbb{N}}$ such that $\forall n\, \beta(A_n, B_n)$ holds.

Define $f_i : X_i \to \{0, \ldots, i + 1\}$ as follows: For $j \leq i$, $f_i(y) = j$ if $\langle a_0, \ldots, a_j \rangle\ R\ \langle b_0, \ldots, b_j \rangle$ fails but $\langle a_0, \ldots a_{j'} \rangle\ R\ \langle b_0, \ldots b_{j'} \rangle$ holds for all $j' < j$. $f_i(y) = i + 1$ if $\langle a_0, \ldots, a_{i'} \rangle\ R\ \langle b_0, \ldots, b_{i'} \rangle$ holds for all $i' \leq i$. (The b_j's here are such that $B_y = \{(0, t_0, b_0), \ldots, (r, t_r, b_r)\}$, and the a_j's are from Alice's above-mentioned request.)

First note that $\operatorname{dom} f_i = X_i$ for all i, since all possible sequences B_y of length $\geq i + 1$ are assigned some value $f_i(y)$ depending on their properties.

If $j \leq i$, then $f_{i+1}^{-1}\{j\} = \{y : y \in X_{i+1} \ \wedge\ \langle a_0, \ldots, a_j \rangle\ R\ \langle b_0, \ldots, b_j \rangle$ fails for the first time at $j\} = f_i^{-1}\{j\} \cap X_{i+1}$.

Finally, $f_i^{-1}\{i\} \subseteq X_{i+1}$, for if $f_i(y) = i$, then $\langle a_0, \ldots, a_i \rangle\ R\ \langle b_0, \ldots, b_i \rangle$ fails for the first time at i. If it happened that $y \notin X_{i+1}$, then $\langle b_0, \ldots, b_i \rangle$ would be a winning response to $\langle a_0, \ldots, a_i \rangle$, which it clearly is not. So $y \in X_{i+1}$.

By the hypothesis that $\langle X_0, \ldots X_{k-1} \rangle$ is a good-for-uniform k-tuple, we know that there exists an element $y \in X_{k-1} \cap f_{k-2}^{-1}\{k-1\}$. So in B_y, $\langle a_0, \ldots, a_{k-2} \rangle\ R\ \langle b_0, \ldots, b_{k-2} \rangle$ holds and in fact $\langle a_0, \ldots, a_j \rangle\ R\ \langle b_0 \ldots, b_j \rangle$ holds for all $j \leq k - 1$, contradicting that (A, B, R') is not k-solvable.

$\square$

Corollary 3.3.14 ($\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$)**.**

(1) Let $k \in \mathbb{N}$. If (A, B, R') is not k-solvable and $\mathsf{SeqP}_k(A, B, R)$ holds, then ACA_0 holds.

(2) If $k < \omega$ is standard, then (1) can be proven in RCA_0.

Proof. This follows from Theorem 3.3.11, Corollary 3.3.12, and Theorem 3.3.13.

□

3.3.5 $\mathsf{I}\Sigma^0_2$ and nonstandard-length good tuples

In the previous subsection, we showed that $\mathsf{I}\Sigma^0_2$ suffices to prove that the nonexistence of a good-for-uniform m-tuple implies ACA_0. At the moment we do not know whether $\mathsf{I}\Sigma^0_2$ is also necessary to prove this. However, in this section we show that $\mathsf{I}\Sigma^0_2$ is necessary to prove the corresponding statement for good tuples: Schmerl's original, non-uniform concept that we introduced in Definition 3.3.5.

Schmerl proved (Lemma 3.3.6) that ACA_0 holds if and only if there are no good m-tuples for standard m; however, extending his proof to nonstandard m appears to require Π^1_1-induction, since we require each $Z_{i-1} = X_{i-1} \setminus (Y_1 \cup \cdots \cup Y_i)$ to be an enumerable set. In our definition of good-for-uniform tuples, our functions f_i enumerate the Y_i's and Z_{i-1}'s uniformly, so this accounts for the difference in the complexity. We now show that when our tuple has nonstandard length, if we are in a model of $\mathsf{B}\Sigma^0_2$, the existence of a good tuple is *equivalent* to $\mathsf{I}\Sigma^0_2 + \neg\mathsf{ACA}_0$.

Definition 3.3.15. A *cut* in a model $\mathcal{N}$ of PA$^-$ is a set $I \subseteq \mathbb{N}$ such that $\forall n \forall m[(n \in I \wedge m < n) \to m \in I]$ and $\forall n(n \in I \to n + 1 \in I)$. A cut $I \subseteq \mathbb{N}$ is called *proper* if $I \neq \varnothing$ and $I \neq \mathbb{N}$.

If ϕ is a formula for which the induction axiom fails in $\mathbb{N}$, then $\psi(n) = (\forall m < n)\phi(m)$ defines a proper cut in $\mathcal{N}$. Similarly, if ϕ defines a proper cut in $\mathbb{N}$, then the induction axiom fails for ϕ.

We reproduce the proof of the following result of Friedman, which is often a useful tool when proving theorems in subsystems with limited induction.

Lemma 3.3.16 (Friedman [8]). *If $\mathcal{N} \models \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$, then there are a proper Σ^0_2 cut $I \subseteq \mathbb{N}$ and an increasing cofinal $c : I \to \mathbb{N}$ whose graph is Δ^0_2.*

Proof. Let $\phi(n) = \exists m\theta(n, m)$ be a Σ^0_2 formula witnessing the failure of $\mathsf{I}\Sigma^0_2$, so that

$$\phi(0) \wedge \forall n(\phi(n) \to \phi(n+1)) \wedge \exists n\neg\phi(n)$$

holds. Let $I = \{n : (\forall k < n)\,\phi(k)\}$. I is clearly a proper cut. We have $n \in I \leftrightarrow (\forall k < n)\exists m\theta(k, m)$, and so by $\mathsf{B}\Sigma^0_2$, $\exists b(\forall k < n)(\exists m < b)\theta(k, m)$, making I a Σ^0_2 cut. Define $c : I \to \mathbb{N}$ by $c(n) = \mu m(\forall k < n)\theta(k, m)$; notice that c is increasing. $\theta(n, m)$ is Π^0_1, so it can be rewritten $\theta(n, m) = \forall k\psi(n, m, k)$ where $\psi(n, m, k)$ is bounded. The graph of c is Δ^0_2, as it is the limit of the graphs of $c_s : I \to \mathbb{N}$ defined by

$$c_s(n) = \mu m\left((\forall k \leq s)\psi(n, m, k) \;\vee\; (m = s)\right)$$

Clearly c_s is computable for all s.

The domain of c is I, for $\mathsf{I}\Sigma^0_1$ is enough to show that if there is an m such that $\theta(n, m)$, there is always a least such m. To show that c is cofinal, if $\exists b(\forall n \in I)(c(n) < b)$, then $\forall n(n \in I \leftrightarrow (\exists m < b)\theta(n, m))$. This means that the cut witnesses the failure of $\mathsf{I}\Pi^0_1$, which is impossible; therefore, c is cofinal.

If c is not increasing, define $c : I \to \mathbb{N}$ by $c(n) = \mu m(\forall k < n)\theta(n, m)$; this increasing cofinal function exists by $\mathsf{B}\Sigma^0_2$.

$\square$

Theorem 3.3.17 (Dorais-Harris, $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$)**.** *Assume that $\mathcal{N} = (\mathbb{N}, \mathfrak{X})$ is a model of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$. Then there exists $m \in \mathbb{N}$ such that there are no good m-tuples.*

Proof. By Lemma 3.3.16, let $I \subseteq \mathbb{N}$ be a proper Σ^0_2 cut, and let $c : I \to \mathbb{N}$ be an increasing cofinal function whose graph is Δ^0_2. Assume that $c(0) = 0$. Also as in Lemma 3.3.16, let $c_s(n) : \mathbb{N} \to \mathbb{N}$ be defined as $c_s(n) = \mu m\left((\forall k \leq s)\psi(n, m, k) \vee (m = s)\right)$, so that $c = \lim_s (c_s \upharpoonright I)$; then $c_s(n)$ is nondecreasing in n.

Fix $m \notin I$. Assume that $\langle X_0, \ldots, X_{m-1}\rangle$ is a good m-tuple. (In this proof, we will define our enumeration functions e_{X_i} to be partial; our proof will be simpler this way.) Assume that if $e_{X_{i+1}}(s) = x$ then $e_{X_i}(s') = x$ for some $s' < s$. Also assume that if $e_{X_i}(s) = x$, then $x < c_s(m-1)$; if not, use a different slower enumeration. This is possible since for $m \notin I$, $\lim_s c_s(m) = \infty$, by the way we have defined c, ψ, and I.

For $1 \leq i \leq m-1$, define Y_i as follows:

$$Y_i = \{x : \exists s (e_{X_i}(s) = x \wedge c_s(i-1) \leq x < c_s(i))\}$$

Clearly $Y_i \subseteq X_i$ and Y_i is enumerable. If $i < j$, then $Y_i \cap Y_j = \varnothing$, for if $x \in Y_i \cap Y_j$, then x enters X_i at an earlier stage than X_j, so $e_{X_i}(s_1) = x \wedge e_{X_j}(s_2) = x$ implies $s_1 < s_2$ and thus $c_{s_1}(i-1) \leq x < c_{s_1}(i) \leq c_{s_2}(i) \leq c_{s_2}(j-1) \leq x < c_{s_2}(j)$, a contradiction.

To show $X_{i-1} \backslash (Y_1 \cup \cdots \cup Y_i)$ is enumerable, we have two cases. Case 1: $i - 1 \in I$. In this case, $Y_1 \cup Y_2 \cup \cdots \cup Y_i \subseteq \{0, \ldots, c(i)\}$, and so $X_{i-1} \cap (Y_1 \cup \cdots \cup Y_i)$ is finite, implying that $X_{i-1} \backslash (Y_1 \cup \cdots \cup Y_i)$ is enumerable. Case 2: $i - 1 \notin I$. In this case, $\lim_s c_s(i-1) = \infty$, and so to determine whether $x \in X_{i-1} \backslash (Y_1 \cup \cdots \cup Y_i)$, we need to check that $e_{X_{i-1}}(s) = x$ for some s, and for this s we need $x \geq c_s(i-1)$; otherwise $x \in Y_1 \cup \cdots \cup Y_{i-1}$. We also need to find the first $t > s$ such that $x < c_t(i-1)$ and ensure that there is no $t' < t$ such that $e_{X_i}(t') = x$ and $c_{t'}(i-1) \leq x < c_{t'}(i)$;

otherwise $x \in Y_i$. This can be checked computably once we know $x \in X_{i-1}$, and so in our Case 2 we have $X_{i-1} \backslash (Y_1 \cup \cdots \cup Y_i)$ is enumerable.

Observe that no obvious uniform sequence of functions exists, since while $X_{i-1} \backslash (Y_1 \cup \cdots \cup Y_i)$ is enumerable, it requires checking whether $i \in I$, and this cannot be done with an obvious $f_{i-1} : X_{i-1} \to \{0, \ldots, i\}$ in our model.

On the other hand, $X_{m-1} \backslash (Y_1 \cup \cdots \cup Y_{m-1}) = \varnothing$. Let $x \in X_{m-1}$; then let s be such that $e_{X_{m-1}}(s) = x$. We know that $x < c_s(m-1)$. Since c_s is nondecreasing and $c_s(0) = 0$, choose i such that $c_s(i-1) \leq x < c_s(i)$ and $x \in Y_i$. Therefore, $\langle X_0, \ldots, X_{m-1} \rangle$ is not actually a good tuple, a contradiction.

$\square$

Corollary 3.3.18 ($\mathsf{RCA_0} + \mathsf{B}\Sigma^0_2$)**.**

$$[\forall m(\ \textit{[there are no good m-tuples]} \to \mathsf{ACA_0}\,)\,] \ \to \ \mathsf{I}\Sigma^0_2$$

.

Proof. This is immediate from Theorem 3.3.17, and the fact that $\neg\mathsf{I}\Sigma^0_2$ implies $\neg\mathsf{ACA_0}$.

$\square$

3.4 Extension: Improved results on Grundy colorings

Schmerl [34] investigated the reverse mathematics of Grundy colorings, and in the process he introduced the concept of a good tuple from Section 3.3. The major results in [34] required certain parameters n to be standard. In this section, we remove that requirement and generalize Schmerl's results.

Definition 3.4.1. Let $G = (V, E)$ be a graph. Let $k \in \mathbb{N}$ and $\varphi : V \to k$ be a coloring. The coloring is *proper* if $\varphi(x) \neq \varphi(y)$ whenever $(x, y) \in E$. The coloring is *Grundy* if for every vertex x,

$$\varphi(x) = \min\{i \in \mathbb{N} : \forall y[(x, y) \in E \to \varphi(y) \neq i]\}$$

Definition 3.4.2. A graph is k-colorable if it has a proper k-coloring. The chromatic number $\chi(G)$ is the smallest such k such that G is k-colorable. The Grundy number $\Gamma(G)$ is the largest possible k such that G has a Grundy k-coloring.

Essentially, a Grundy coloring orders the vertices in some way, and applies a greedy coloring algorithm, coloring each vertex in turn with the smallest possible color. Different orderings of the vertices will result in different Grundy colorings from different greedy algorithms; $\Gamma(G)$ is the maximum possible number of colors.

Figure 3.1 gives an example: two identical crown graphs with different vertex labels; with these labels, the greedy algorithm will produce the two distinct Grundy colorings given in the figure. Here the vertex name is given by the number label, and the color given by the shading. The colors are ordered from lightest to darkest (hence in the right graph: white=1, red=2, blue=3, black=4).

We can also define $\gamma(G)$ as the smallest possible k such that G has a Grundy k-coloring. If we assume all of second-order arithmetic, we can show that $\chi(G) = \gamma(G)$: Within the collection of minimal colorings, choose the one with a maximal set of vertices for color 1; given that set, choose a maximal set of vertices for color 2, and so on. This color is clearly Grundy. If we have Zorn's Lemma, we can extend this to infinite graphs as well. However, when working in weaker subsystems, $\chi(G)$ and $\gamma(G)$ may well be unequal.

Definition 3.4.3. Let $\mathcal{F}$ be a class of finite graphs. Forb($\mathcal{F}$) is the class of countable

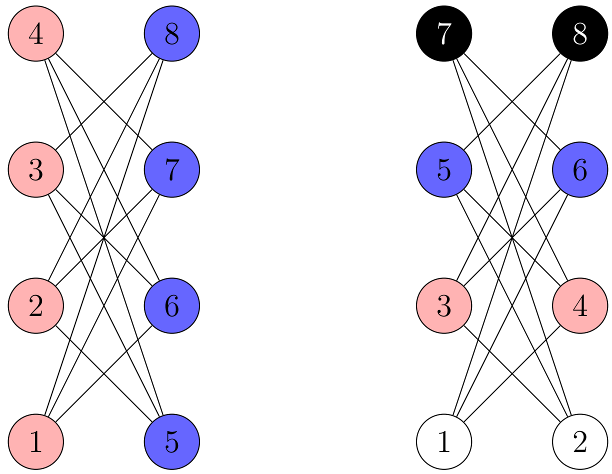

Figure 3.1: Two different Grundy colorings of a crown graph G; the right one witnesses $\Gamma(G) = 4$.

graphs that do not have an induced subgraph isomorphic to any graph in $\mathcal{F}$.

Definition 3.4.4. $\mathcal{C}$ is a *natural class* of graphs if $\mathcal{C} = \text{Forb}(\mathcal{F})$ for some set of connected finite graphs $\mathcal{F}$.

Definition 3.4.5. Let $\mathcal{C}$ be a natural class of graphs.

- $\chi(\mathcal{C}) = \sup\{\chi(G) : G \in \mathcal{C}\}$
- $\gamma(\mathcal{C}) = \sup\{\gamma(G) : G \in \mathcal{C}\}$
- $\Gamma(\mathcal{C}) = \sup\{\Gamma(G) : G \in \mathcal{C}\}$

It is trivial that as long as all three exist, we have $\chi(\mathcal{C}) \leq \gamma(\mathcal{C}) \leq \Gamma(\mathcal{C})$. However, showing that $\chi(\mathcal{C}) = \gamma(\mathcal{C})$ requires ACA_0:

Theorem 3.4.6 (Schmerl, Theorem 1.1 in [34]) **(RCA_0).** *Let $n < \omega$. If there is a natural class $\mathcal{C}$ such that $\gamma(\mathcal{C}) < n = \Gamma(\mathcal{C})$, then ACA_0 holds.*

Notice that this theorem requires n to be standard. To help us prove that the statement holds when n is not standard, we recall a useful propostition from Schmerl's paper.

Definition 3.4.7. If G is a graph and $W \subseteq V(G)$, then $G[W]$ is the induced subgraph H of G such that $V(H) = W$.

Proposition 3.4.8 (Schmerl, Lemma 2.1 in [34]) ($\mathsf{RCA_0}$). *Suppose that $\varphi : V(G) \to \mathbb{N}$ is a Grundy coloring of the graph G, and suppose that $a \in V(G)$ and $\varphi(a) = x$. Then there is a $W \subseteq V(G)$ such that $a \in W$, $|W| \leq 2^x$, and $\varphi \restriction W$ is a Grundy coloring of $G[W]$.*

Proof. We prove this by induction on x. If $x = 0$, then let $W = \{a\}$. Now suppose that $b \in V(G)$ is such that $\varphi(b) = x + 1$. Since φ is a Grundy coloring, b must connect to x different vertices that are colored with all available colors $\leq x$. For each $y \leq x$, let $a_y \in V(G)$ be the least vertex such that $(a_y, b) \in E(G)$ and $\varphi(a_y) = y$. By the inductive hypothesis, let W_y be such that $a_y \in W_y, |W_y| \leq 2^y$, and $\varphi \restriction W_y$ is a Grundy coloring of $G[W_y]$. Let $W = W_0 \cup W_1 \cup \cdots \cup W_x$ and W has the desired properties.

□

We are ready to prove:

Theorem 3.4.9 ($\mathsf{RCA_0} + \mathsf{I}\Sigma^0_2$). *Suppose that there is a natural class $\mathcal{C}$ such that $\gamma(\mathcal{C}) < \Gamma(\mathcal{C})$. Then $\mathsf{ACA_0}$ holds.*

Proof. We will mimic Schmerl's proof of Theorem 3.4.6, but will use our concept of a good-for-uniform tuple from Section 3.3.

We work in a model $\mathcal{N} = (\mathbb{N}, \mathfrak{X})$ that satisfies $\mathsf{RCA_0}$. Assume that $\mathcal{N} \not\models \mathsf{ACA_0}$. First let us consider the case where $\mathcal{C}$ is a natural class with $\Gamma(\mathcal{C}) = n \in \mathbb{N}$; we will handle the infinite case separately. Note that $n \geq 3$.

By the proof of Proposition 3.4.8, there is $G' = (V', E') \in \mathcal{C}$ such that $\Gamma(G') = n$ and $|V'| \leq 2^{n-1}$. Let $G = (V, E)$ be a minimal induced subgraph of G' such that

$\Gamma(G) = n$ (and since natural classes are closed under finite induced subgraphs, $G \in \mathcal{C}$ as well); for the remainder of the proof we will work with G and not G'. For each $i < n$, let $G_i = (V_i, E_i)$ be the induced subgraph of G consisting of those vertices x for which $\varphi(x) \leq i$. From the proof of Proposition 3.4.8 we see that if $G_i \subsetneq G$, then $\Gamma(G_i) < n$. (Since in the construction, the vertex a_n is the first that we are connecting to n vertices.) Let $\varphi : V \to n$ be a Grundy coloring of G that witnesses $\Gamma(G) = n$.

By Theorem 3.3.11, there exists a good-for-uniform n-tuple $\langle X_0, X_1, \ldots, X_{n-1} \rangle$. We define a graph $H = (W, F)$ that is the disjoint union of graphs H_x, for $x \in X_0$, such that $H_x = (W_x, F_x)$ is isomorphic to G_i if and only if $x \in X_i \backslash X_{i+1}$ (where $X_n = \varnothing$). Precisely, for each $i < n$,

$$(s,\, v,\, x) \in W_x \iff (\exists i < n)(e_{X_i}(s) = x) \,\wedge\, (v \in V_i \backslash V_{i-1})$$

$$((s_1, v_1, x),\, (s_2, v_2, x)) \in F_x \iff (v_1, v_2) \in E$$

Let $h_x : V_i \to W_x$ be an explicit isomorphism between G_i and H_x:

$$h_x(v) = (s, v, x)$$

Let $H = \cup_x H_x = \cup_x (W_x, F_x)$. Crucially, our usage of s such that $e_{X_i}(s) = x$ ensures that our graph $H \in \mathfrak{X}$, even though each X_i is an enumerable set and may not be in $\mathfrak{X}$.

Let $N = \cup_x W_x$, so that N is the set of vertices of H. Clearly $H \in \mathcal{C}$, so since by assumption $\gamma(\mathcal{C}) < n$, we know that there exists a Grundy coloring ψ for H, $\psi : N \to (n-1)$. The coloring ψ exists in our model, i.e., $\psi \in \mathfrak{X}$.

Define our sequence $\langle f_0, \ldots, f_{n-2} \rangle$, $f_i : X_i \to \{0, 1, \ldots, i+1\}$, as follows:

$$f_i(x) = j \leq i \iff \psi \circ h_x \restriction V_j \neq \varphi \restriction V_j \text{ and this is the first } j \text{ where this fails.}$$
$$f_i(x) = i+1 \iff \psi \circ h_x \restriction V_j = \varphi \restriction V_j \text{ for all } j \leq i$$

Let us check that our f_i's satisfy the hypotheses for a good-for-uniform n-tuple. Clearly $\operatorname{dom} f_i = X_i$. It is also clear that $f_{i+1}^{-1}\{j\} = f_i^{-1}\{j\} \cap X_{i+1}$ for all $j \leq i < n-2$. We need to check that $f_i^{-1}\{i\} \subseteq X_{i+1}$ for all $i < n-1$. For $i = 0$, if $f_0(x) = 0$ and $x \in X_0 \backslash X_1$, then since φ is Grundy, W_x is a set of disconnected points, and so φ and $\psi \circ h_x$ are both identically 0 on V_0, since both are Grundy. Now suppose $0 < i < n-1$ and $f_i(x) = i$, so that $\psi \circ h_x \restriction V_{i-1} = \varphi \restriction V_{i-1}$. If we also had $x \in X_i \backslash X_{i+1}$, then H_x is isomorphic to G_i, and we have $\psi \circ h_x = \varphi \restriction V_i$, since ψ and φ are both Grundy, and the only vertices in $V_i \backslash V_{i-1}$ are those with $\varphi(x)$ exactly equal to i, while all vertices in V_{i-1} are already colored according to φ. This of course is a contradiction, since it means $f_i(x) = i+1$, and so our assumption that $x \in X_i \backslash X_{i+1}$ was wrong.

So all the hypotheses are satisfied, and we can conclude that $f_{n-2}^{-1}\{n-1\} \cap X_{n-1} \neq \varnothing$. That is, there is $x \in X_{n-1}$ with $\psi \circ h_x \restriction V_{n-2} = \varphi \restriction V_{n-2}$. So $H_x \cong G_{n-1}$; by a similar argument as in the last paragraph, since ψ is Grundy, we also have $\psi \circ h_x = \varphi \restriction V_{n-1} = \varphi$. But this is a contradiction since $n-1 \in \operatorname{ran}(\varphi)$ but $n-1 \notin \operatorname{ran}(\psi)$. So ACA_0 holds.

Finally, what if $\gamma(\mathcal{C}) < \Gamma(\mathcal{C}) = \infty$? We can still use the proof of Proposition 3.4.8. Let $G \in \mathcal{C}$ be a graph with $\Gamma(G) \geq \gamma(\mathcal{C})+1$, and let $\varphi : V(G) \to \mathbb{N}$ be a coloring that witnesses this inequality. Let x in that proof be $\gamma(\mathcal{C})+1$, and let $a \in V(G)$ be such that $\varphi(a) = x$. Then there is $H \subseteq G$ such that $a \in V(H)$, $|V(H)| \leq 2^x$, and $\varphi \restriction V(H)$ is a Grundy coloring of H. Thus $H \in \mathcal{C}$ is a finite graph with $\gamma(H) < \Gamma(H) = \gamma(\mathcal{C})+1$ and we can proceed as in the first part of the proof.

$\square$

As a corollary, we can get Theorem 3.4.6, which states that Theorem 3.4.9 is provable in $\mathsf{RCA_0}$ if $\gamma(\mathcal{C})$ is standard. Just replace the reference to Theorem 3.3.11 with a reference to Corollary 3.3.12 in the proof of Theorem 3.4.9.

3.5 Conclusion (Metatheorem)

Metatheorem 3.5.1. *Let $\mathsf{P} = (A, B, R)$ be a bounded problem with a primitive recursive relation R. (A, B, R') is the closed kernel of P.*

(a) If (A, B, R) is on-line solvable, then $\mathsf{RCA_0} \vdash \mathsf{SeqP}_k(A, B, R)$ for every standard $k < \omega$.

(b) If (A, B, R') is solvable but (A, B, R) is not on-line solvable, then there exists a standard $k < \omega$ such that $\mathsf{RCA_0} \vdash (\mathsf{SeqP}_k(A, B, R) \leftrightarrow \mathsf{WKL_0})$.

(c) If (A, B, R) is solvable but (A, B, R') is not solvable, then there exists a standard $k < \omega$ such that $\mathsf{RCA_0} \vdash (\mathsf{SeqP}_k(A, B, R) \leftrightarrow \mathsf{ACA_0})$.

Proof. First we prove (a). Let $k < \omega$. Bob has a winning strategy in $\mathsf{G}_k(A, B, R)$; since R is primitive recursive, $k < \omega$, and the problem is bounded, the maximum play $M_k(A, B, R)$ exists and is standard. (Recall Definition 3.2.3.) Therefore, Bob's entire strategy in $\mathsf{G}_k(A, B, R)$ is primitive recursive, and so $\mathsf{RCA_0}$ proves that Bob has a winning strategy in $\mathsf{G}_k(A, B, R)$. Now we can apply Theorem 3.2.2 to show that $\mathsf{RCA_0} \vdash \mathsf{SeqP}_k(A, B, R)$.

Next we prove (b). Let $k < \omega$. If $\bar{a}$ is a request of length k by Alice, then Bob has a winning response $\bar{b}$ such that $(\bar{a} \restriction j)\ R\ (\bar{b} \restriction j)$ holds for all j, $0 < j \leq k$. Since R is primitive recursive, $k < \omega$, and the problem is bounded, the maximum play $M_k(A, B, R)$ exists and is standard. Therefore, in $\mathsf{RCA_0}$ we can check all possible $\bar{a}$ of length $\leq k$ and prove that there is indeed a $\bar{b}$ of length $\leq k$ such that $(\bar{a} \restriction j)\ R\ (\bar{b} \restriction j)$

holds for all j, $0 < j \leq k$. Therefore, RCA_0 proves that $(A \cap \mathbb{N}^{<k}, B \cap \mathbb{N}^{<k}, R' \cap (\mathbb{N}^{<k} \otimes \mathbb{N}^{<k}))$ is solvable. Now we can apply Theorem 3.3.2 to show that WKL_0 proves $\mathsf{SeqP}_k(A, B, R)$.

For the other direction, if (A, B, R) is not on-line solvable, then we can essentially prove versions of Lemma 3.2.7 and Theorem 3.2.8 using full second-order arithmetic. The major difference is that the "$k \in \mathbb{N}$" in the hypothesis of Lemma 3.2.7 can be replaced with "$k < \omega$"; similarly, our version of Theorem 3.2.8 will prove the existence of a $k < \omega$ such that Alice has a winning strategy in $\mathsf{G}_k(A, B, R)$ if the game $\mathsf{G}(A, B, R)$ is not on-line solvable. Since the game is bounded, we can show that for this k, $M_k(A, B, R)$ exists and is standard. Since we also have that R is primitive recursive, we can prove in RCA_0 that Alice has a winning strategy in $\mathsf{G}_k(A, B, R)$. By Corollary 3.2.6, we can thus prove in RCA_0 that $\mathsf{SeqP}_k(A, B, R) \to \mathsf{WKL}_0$.

Finally we prove (c). Let $k < \omega$. By definition, since (A, B, R) is solvable, for every request $\bar{a}$ of length k, there exists a response $\bar{b}$ such that $\bar{a}\ R\ \bar{b}$ holds. Since R is primitive recursive, $k < \omega$, and the problem is bounded, the maximum play $M_k(A, B, R)$ exists and is standard. Therefore, in RCA_0 we can check all possible $\bar{a}$ of length $\leq k$ and prove that there is indeed a $\bar{b}$ of length $\leq k$ such that $\bar{a}\ R\ \bar{b}$. Therefore, RCA_0 proves that $(A \cap \mathbb{N}^{<k}, B \cap \mathbb{N}^{<k}, R' \cap (\mathbb{N}^{<k} \otimes \mathbb{N}^{<k}))$ is solvable. Now we can apply Proposition 3.3.1 to show that ACA_0 proves $\mathsf{SeqP}_k(A, B, R)$.

For the other direction, if (A, B, R') is not solvable, then there exists a play $\bar{a}$ of some standard length k such that for any $\bar{b}$ of length k, we have $(\bar{a} \restriction j)\ R\ (\bar{b} \restriction j)$ fails for some $j \leq k$. Since the problem is bounded and R is primitive recursive, we can prove this in RCA_0: simply enumerate all responses $\bar{b}$ whose plays are bounded by $M_k(A, B, R)$, and show that the above statement applies to each such $\bar{b}$. Therefore, RCA_0 proves that (A, B, R') is not solvable. By part (2) of Corollary 3.3.14, we can prove in RCA_0 that $\mathsf{SeqP}_k(A, B, R) \to \mathsf{ACA}_0$.

$\square$

Recall our Conjecture 2.3.20 that $\exists k \mathsf{Evade}_k(2)$ may be strictly weaker than $\exists r \mathsf{Evade}_1(r)$, which *is* strictly weaker than $\mathsf{WKL_0}$ in models of $\neg \mathsf{I}\Sigma^0_2$. If we do not have access to a standard $k < \omega$ as in the hypothesis of Corollary 3.2.6, our result must involve this weaker principle.

Theorem 3.5.2. *Let (A,B,R) be a bounded problem which is not on-line solvable. Then* $\mathsf{RCA_0} \vdash \big(\mathsf{SeqP}(A, B, R) \to \exists k \exists r \mathsf{Evade}_k(r)\big)$.

Proof. Using the proof of the second direction of part (b) of the Metatheorem, we can show that there is a k such that $\mathsf{RCA_0}$ proves that Alice has a winning strategy in $\mathsf{G}_k(A, B, R)$. Then we can apply Theorem 3.2.5 to show, in $\mathsf{RCA_0}$, that if $\mathsf{SeqP}_k(A, B, R)$ holds, then there exists M such that $\mathsf{Predict}_k(M)$ fails. Take $r = M$ and $\mathsf{Evade}_k(r)$ holds.

$\square$

Chapter 4

Applications

In Chapter 3 we classified all sequential problems by reverse-mathematical strength. In Chapter 4, we will apply these results to show the reverse-mathematical strength of a variety of concrete sequential problems.

We have chosen to focus on:

- **Combinatorial problems**: Pigeonhole principles (4.1), Dilworth's theorems (4.7), Ramsey's theorems (4.8).
- **Classic on-line algorithm problems**: Task scheduling problem (4.4), Paging problem (4.5), List update problem (4.6).
- **Both of the above**: Graph colorings (4.2), Marriage/matching problems (4.3).
- **Purely reverse-mathematical questions**: Δ^0_2–evasion (last part of 4.1), Separating sets (4.9).

Most central ideas are presented in sections 4.1 through 4.4, which is why we present them first.

Each section will include a definition of the form, "Let P be the problem (A, B, R)," where A, B are trees and R is a relation. This comes from our formal definition of "problem" presented in Section 3.1, whose definitions we will refer to throughout this chapter. Recall that ACA_0 is sufficient to prove the sequential version of any solvable problem; this is Proposition 3.3.1 and will not always be cited explicitly.

In some of these sections (4.3, 4.8, 4.9) we also consider *unbounded* problems. These will essentially always require ACA_0, but there is some nuance to their proofs.

In the concluding Section 4.10, we will summarize all applications we have considered, classifying them by their proof-theoretic strength. We will also revisit the original motivation for investigating sequential problems — determining whether the nonsequential problem requires the Law of Excluded Middle.

4.1 Pigeonhole Principles

The Pigeonhole Principle is a prototypical example of a problem without a solvable closed kernel, and thus ACA_0 is required to prove the sequential version. The non-optimal thin variation, presented in Theorem 4.1.2, is equivalent to ACA_0 over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$. The proofs below provide a good overview of the "all solutions have an unsolvable initial segment" condition and the ways we might encounter it, particularly the non-optimal thin variation, where our example has $k-1$ distinct correct answers, all of which fail at different initial segments.

Theorem 4.1.1. *The following are equivalent over* RCA_0*:*

(i) ACA_0

(ii) The Sequential Finite Pigeonhole Principle: Given $k \geq 2$ *and a sequence* $\langle A_n, f_n \rangle_{n\in\mathbb{N}}$*, where* A_n *is a finite set and* $f_n : A_n \to \{0, 1, \ldots, k-1\}$*, there is a sequence* $\langle y_n \rangle_{n\in\mathbb{N}}$ *such that*

$$\forall n \left(|\{x \in A_n : f_n(x) = y_n\}| \geq \frac{|A_n|}{k} \right).$$

(iii) *The Non-Optimal Sequential Finite Pigeonhole Principle: Let* $k \geq 2$ *and* $p \in (0, \frac{1}{k}]$. *Given a sequence* $\langle A_n, f_n \rangle_{n \in \mathbb{N}}$, *where* A_n *is a finite set and* $f_n : A_n \to \{0, \ldots, k-1\}$, *there is a sequence* $\langle y_n \rangle_{n \in \mathbb{N}}$ *such that*

$$\forall n \ \left(|\{x \in A_n : f_n(x) = y_n\}| \ \geq \ p|A_n| \right).$$

Proof. It is clearly enough to show that (i) $\Leftrightarrow$ (iii). (Note that (iii) is false if $p > \frac{1}{k}$.)

Define the problem $\mathsf{PP}(k, p)$ as the following problem (A, B, R): $A = k^{<\infty}$; B is the tree of all constant finite sequences of elements in $\{0, \ldots, k-1\}$; R is defined by: $\bar{a} \ R \ \bar{b}$ if and only if $\bar{b} = \langle y, \ y, \ \ldots, \ y \rangle$ and $|\{i : a_i = y\}| \geq p|\bar{a}|$.

It is easy to see that $\mathsf{SeqPP}(k, p)$ and (iii) are equivalent over $\mathsf{RCA_0}$. There is a direct correspondence between triples $(i, s_i, a_i) \in X_n$ in the hypothesis of $\mathsf{SeqPP}(k, p)$ and pairs $(x, f_n(x))$ in the hypothesis of (iii); the rest of the equivalence is straightforward. Notice that this equivalence would fail if we only required the *last* response in $\bar{b}$ to be the correct y. For if we assumed $\mathsf{SeqPP}(k, p)$ and the hypothesis of (iii), we would not know how long to wait until y was enumerated.

Let N be such that $N > 1/p$. Note that this implies that $N > k$.

Let $\bar{a}$ be defined as follows:

$$\begin{aligned}
a_0 &= 0 \\
a_1 = \cdots = a_N &= 1 \\
a_{N+1} = \cdots = a_{2N} &= 2 \\
&\cdots \\
a_{(k-1)(N-1)+1} = \cdots = a_{(k-1)N} &= k-1 \\
a_{(k-1)N+1} = \cdots = a_{(k-1)N+N^2-1} &= 0
\end{aligned}$$

The only correct response is $\bar{b} = \bar{0}$, for $|i : a_i = 0| = N^2 \geq \frac{2}{k}N^2 \geq 2pN^2 = pN(2N) \geq pN(k-1+N) = p((k-1)N + N^2) = p|\bar{a}|$.

For any other $j < k$, we have $|i : a_i = j| = N < k-1+N < pN(k-1+N) = p((k-1)N + N^2) = p|\bar{a}|$.

However, if we consider the initial segment $(\bar{a} \upharpoonright (k-1)N+1)$, 0 is not a correct response, since 0 only occurs once as a request, and the initial segment has length $(k-1)N + 1 \geq N > 1/p$.

Therefore, the closed kernel of this problem is not solvable, and so $\mathsf{SeqPP}(k, p)$ and (iii) both are equivalent to ACA_0 by Proposition 3.3.3.

□

In the above example, I choose a problem where both the full $\bar{a}$ and the initial segment $\bar{a} \upharpoonright j$ contained every $i < k$; there are simpler examples where this is not the case.

The next example asserts the existence of a sequence of "least popular" choices, choosing an element $y < k$ such that at least $(1 - \frac{1}{k})$ of the domain gets assigned an element other than y:

Theorem 4.1.2. *The following are equivalent over* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$*:*

(i) ACA_0

(ii) *The Sequential Pigeonhole Principle, Thin Variation: Given $k \geq 2$ and a sequence $\langle A_n, f_n \rangle_{n\in\mathbb{N}}$, where A_n is a finite set and $f_n : A_n \to \{0, 1, \ldots, k-1\}$, there is a sequence $\langle y_n \rangle_{n\in\mathbb{N}}$ such that*

$$\forall n\ (|\{x \in A_n : f_n(x) \neq y_n\}| \ \geq \ \frac{|A_n|(k-1)}{k}).$$

(iii) *The Sequential Pigeonhole Principle, Non-Optimal Thin Variation : Let $k \geq 2$ and $p \in (0,\, 1 - \frac{1}{k}]$. Given a sequence $\langle A_n, f_n \rangle_{n\in\mathbb{N}}$, where A_n is a finite set and $f_n : A_n \to \{0, \ldots, k-1\}$, there is a sequence $\langle y_n \rangle_{n\in\mathbb{N}}$ such that*

$$\forall n\ (|\{x \in A_n : f_n(x) \neq y_n\}| \ \geq \ p|A_n|.$$

Proof. It is clearly enough to show that (i) $\Leftrightarrow$ (iii). Note that (iii) is false if $p > 1 - \frac{1}{k}$.

In proving this example, it is particularly evident why we need the results about the closed kernel from the previous chapter. In Theorem 2.2.1, we saw a straightforward (not using closed kernels) proof for the optimal problem, but the non-optimal problem is tricky for $p \leq 1 - \frac{2}{k+1}$. There are often multiple correct choices for y_n, and for p small enough, there is at most one *wrong* choice for y_n.

Define the problem $\mathsf{ThinPP}(k, p)$ as the following problem (A, B, R): $A = k^{<\infty}$; B is the tree of all constant finite sequences of elements of $\{0, \ldots, k-1\}$; R is defined by: $\bar{a}\ R\ \bar{b}$ if and only if $\bar{b} = \langle y,\, y,\, \ldots,\, y \rangle$ and $|\{i : a_i \neq y\} \geq p|\bar{a}|$. It is easy to see that $\mathsf{SeqThinPP}(k, p)$ and (iii) are equivalent over $\mathsf{RCA_0}$.

Let N be such that $N > \max(1/p, k)$.

Let $\bar{a}$ be defined as follows:

$$\begin{aligned}
a_0 &= 0 \\
a_1 = \cdots = a_N &= 1 \\
a_{N+1} = \cdots = a_{N+N^2} &= 2 \\
a_{N+N^2+1} = \cdots = a_{N+N^2+N^3} &= 3 \\
&\cdots \\
a_{N+N^2+\cdots+N^{k-2}+1} = \cdots = a_{N+N^2+\cdots+N^{k-1}} &= k-1 \\
a_{N+N^2+\cdots+N^{k-1}+1} = \cdots = a_{N+N^2+\cdots+N^{k-1}+N^k} &= 0
\end{aligned}$$

Any constant response is correct other than $\bar{b} = \bar{0}$ which is incorrect, since $|i : a_i \neq 0| = N + N^2 + \cdots + N^{k-1} < pN(N + N^2 + \cdots + N^{k-1}) = p(N^2 + \cdots + N^k) < p(1 + N + N^2 + \cdots + N^k) = p|\bar{a}|$. Observe that if it happened that $a_0 \neq 0$, a very similar argument would still hold.

Let us be sure that $\bar{b} = \overline{k-1}$ is a correct response; if it is, then $\bar{1}, \ldots, \overline{k-2}$ certainly are. Note that $N > k$ implies that $N^k > N^{k-1}(k-1)$ which implies that $N^{k-1} - (N^k + N^{k-1})/k < 0$. From this we can conclude that $|i : a_i \neq k-1| = 1 + N + N^2 + \cdots + N^{k-2} + N^k > 1 + N + N^2 + \cdots + N^{k-2} + (1 - 1/k)(N^k + N^{k-1}) \geq p(1 + N + N^2 + \cdots + N^{k-1} + N^k) = p|\bar{a}|$.

So we have $k-1$ correct responses, the constant sequences $\bar{1}, \bar{2}, \ldots, \overline{k-1}$, and one incorrect response $\bar{0}$. However, *every single one* of the correct responses is incorrect at some initial segment: namely, $\bar{i}$ is incorrect at $\bar{a} \upharpoonright (1 + N + N^2 + \cdots + N^i)$; see our observation in the second-to-last-paragraph.

Therefore, the closed kernel of this problem is not solvable, and so $\mathsf{SeqThinPP}(k, p)$ and (iii) both are equivalent to ACA_0 over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$ by Corollary 3.3.14.

□

Δ^0_2–Evasion

In Section 2.3, we introduced the equivalent principles $\mathsf{DNR}(k)$ and $\mathsf{Evade}_1(k)$ (both equivalent to $\mathsf{WKL_0}$ for standard $k \geq 2$), which assert the existence of a function $g : \mathbb{N} \to k$ that avoids every Σ^0_1-function $f \subseteq: \mathbb{N} \to k$. Here we consider a similar principle for Δ^0_2 functions, which informally asserts that for every Δ^0_2 function $f : \mathbb{N} \to k$, there exists an evading function $g : \mathbb{N} \to k$ such that $\forall n(g(n) \neq f(n))$. Because of the relationship between this principle and the non-optimal pigeonhole principle, we include it in this section.

Definition 4.1.3. Let $k \geq 3$. Δ^0_2–$\mathsf{Evade}_1(k)$ is the statement: Let $\langle f_s \rangle_{s\in\mathbb{N}}$ be a sequence of functions, $f_s : \mathbb{N} \to k$ such that $\forall n\, \exists t(\forall t' > t)(f_{t'}(n) = f_t(n))$. Then there exists a function $g : \mathbb{N} \to k$ such that $\forall n(g(n) \neq f_t(n))$, where t is the witness in the above hypothesis.

We start by considering $k = 3$ because Δ^0_2–$\mathsf{Evade}_1(2)$ very clearly allows us to compute the halting problem, and is therefore equivalent to $\mathsf{ACA_0}$. For $k \geq 3$, this equivalence is less immediate. Nevertheless, by observing a relationship with the non-optimal pigeonhole principle, we show that it too is equivalent to $\mathsf{ACA_0}$.

Proposition 4.1.4 ($\mathsf{RCA_0}$)**.** *For $k \geq 3$, the following are equivalent:*

(i) Δ^0_2–$\mathsf{Evade}_1(k)$

(ii) The Sequential Pigeonhole Principle, Non-Optimal Thin Variation for $p \in (0, 1/2]$: Let $p \in (0, 1/2]$. Assume that $\langle A_n, f_n \rangle_{n\in\mathbb{N}}$ are such that A_n is a finite set and $f_n : A_n \to \{0, \dots, k-1\}$. Then there is a sequence $\langle y_n \rangle_{n\in\mathbb{N}}$ such that

$$\forall n \ \left(|\{x \in A_n : f_n(x) \neq y_n\}| \ \geq \ p|A_n|\right).$$

Proof. (i) $\Rightarrow$ (ii): Assume Δ^0_2–$\mathsf{Evade}_1(k)$, and let $\langle A_n, f_n\rangle_{n\in\mathbb{N}}$ be as in the hypothesis of (ii). Define $\langle h_s\rangle_{s\in\mathbb{N}}$, $h_s : \mathbb{N} \to k$, as follows: $h_s(n) =$ the least j such that

$$(\forall j' < k)\ |\{x : x \le s \wedge f_n(x) = j'\}| \le |\{x : x \le s \wedge f_n(x) = j\}|.$$

Since the domain of each f_n is finite, there clearly exists a limiting $t(n)$ as in the hypothesis of Δ^0_2–$\mathsf{Evade}_1(k)$. Let g be the evading function guaranteed by (i). Define $y_n = g(n)$. Then for a given n, $|\{x \in A_n : f_n(x) \ne y_n\}| \ge \frac{1}{2} \cdot |A_n| \ge p|A_n|$, with the first inequality being an equality only when $|\operatorname{ran} f_n| = 2$ and the two values occur equally often, with $g(n)$ choosing the higher of the two.

(ii) $\Rightarrow$ (i): Assume (ii), and let $\langle f_s\rangle_{s\in\mathbb{N}}$ be functions that satisfy the hypothesis in (i). Also, let $N > 1/p$. Define a sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ of finite functions as follows: For each n, $h_n(0) = f_0(n)$. At stage $s > 0$, let t be the greatest value such that $h_n(t)$ is defined at stage $s-1$, and let M be the total number of values of h_n defined by stage $s-1$. If $f_s(n) = j \ne f_{s-1}(n)$, then define $h_n(t+1) = j, \ldots, h_n(t+NM) = j$. Each h_n will be finite, as the $f_s(n)$'s will eventually reach their limit. Let $\langle y_n\rangle_{n\in\mathbb{N}}$ be the sequence guaranteed by (ii). If $j = \lim_s f_s(n)$, then $|\{x : h_n(x) \ne j\}| \le M = \frac{1}{N}(NM) < p \cdot (NM)$, where M is the final value of M used in the construction. Therefore, $y_n \ne j$. So if we define $g : \mathbb{N} \to k$ by $g(n) = y_n$, then we have $g(n) \ne \lim_s f_s(n)$ for all n, as desired.

□

Corollary 4.1.5 ($\mathsf{RCA_0} + \mathsf{I}\Sigma^0_2$)**.** *For $k \ge 2$, we have* Δ^0_2–$\mathsf{Evade}_1(k) \leftrightarrow \mathsf{ACA_0}$.

4.2 Graph Colorings

Graph colorings have played a major role throughout this thesis. Basic graph-related definitions were given in Section 1.2, and much of the work in this section has already been shown in Section 2.4. However, in this section our Chapter 3 results will allow us to analyze the sequential coloring problem more efficiently.

A finite graph G can be coded as $\langle e_0, e_1, \ldots, e_n \rangle$, with the length n of the sequence giving the number of vertices, and $e_i < 2^i$ encoding through its binary representation how vertex v_i connects by an edge to $v_0, \ldots, v_{n-1}$. In this section, whenever we talk about a set of "coded finite graphs," we will be talking about a set $C \subseteq \mathbb{N}$ of codes as described above.

Definition 4.2.1. Let C be a set of coded finite graphs. The corresponding *universal class* $\mathcal{C}$ consists of all graphs $G = (V, E)$ all of whose finite induced subgraphs are isomorphic to an element of C.

We say that $\mathcal{C}$ is *k-bounded* if every element of C (and hence every element of $\mathcal{C}$) has size at most k.

Given any universal class $\mathcal{C}$, we write $\mathcal{C}_{\leq k}$ for the k-bounded universal class generated by the elements of C with size at most k.

For example, we could take C to be the set of all (coded) finite forests, and the corresponding $\mathcal{C}$ would be the universal class of all forests. Or, as in Proposition 1.2.6, if we wished $\mathcal{C}$ to be the universal class of all bipartite graphs that avoid P_6, then C would be the set of all finite graphs of this type. Though it is not required, we may always assume that the set C is closed under isomorphisms and induced subgraphs.

Since there are only finitely many codes for graphs on n vertices, the universal class $\mathcal{C}$ is always a Π^0_1-class with parameter C. Notice that in the two examples for C

in the previous paragraph, the set C is definable by a primitive recursive predicate.

Definition 4.2.2. $\mathsf{Col}(\mathcal{C}, r)$ is the problem (A, B, R), where $A = \mathbb{N}^{<\infty}, B = r^{<\infty}$. $(\bar{a}, \bar{b}) \in R$ if the following is true: *If* $a_i < 2^i$, and *if* by using the binary representation of a_i to determine which of vertices $0, \ldots, i-1$ are connected to vertex i, *we have that the resulting graph is in* $\mathcal{C}$, then coloring vertices according to $\bar{b}$ results in a valid coloring.

Let us explicitly describe the game $\mathsf{G}(\mathcal{C}, r)$, which is the game $\mathsf{G}(A, B, R)$ using our terminology from Definition 3.1.2:

- Alice plays a new graph vertex and specifies whether or not it is connected by an edge with each vertex that she played on an earlier round. Alice is guaranteed to lose if the graph played thus far does not belong to the class $\mathcal{C}$.

- Bob responds by assigning a color from $\{0, \ldots, r-1\}$ to the vertex that Alice just played. Bob loses immediately if the colors assigned thus far do not form a valid r-coloring of the graph.

Bob wins if the game goes on indefinitely without either player losing.
If we exclude the "invalid" moves that immediately guarantee a player's loss, each player has only finitely many possible moves on each round and the player's moves can be coded using integers of fixed size:

$$\begin{array}{r|cccccccc} \text{Alice} & e_0 & & e_1 & & e_2 & & \cdots & \\ \hline \text{Bob} & & c_0 & & c_1 & & c_2 & & \cdots \end{array}$$

where $e_i < 2^i$ encodes through its binary representation how the i^{th} vertex played by Alice is connected by an edge with the previous vertices, and $c_i < r$ is the color that Bob assigns to the i^{th} vertex.

Our major results in Chapter 3 give the following:

Theorem 4.2.3 ($\mathsf{RCA_0}$)**.** *Let $\mathcal{C}$ be a universal class of graphs such that $\mathsf{Col}(\mathcal{C}, r)$ is solvable. Then $\mathsf{WKL_0} \vdash \mathsf{SeqCol}(\mathcal{C}, r)$.*

Proof. $\mathsf{Col}(\mathcal{C}, r)$ is a semi-bounded problem (responses are bounded by r) with a solvable closed kernel, since every finite restriction of a valid coloring is still a valid coloring. By Theorem 3.3.2, $\mathsf{WKL_0}$ proves $\mathsf{SeqCol}(\mathcal{C}, r)$.

□

The above theorem only required $\mathsf{Col}(\mathcal{C}, r)$ to be semi-bounded, but there is an equivalent *bounded* problem: in round j, Alice simply plays the edge relation with all vertices played in rounds 0 through $j - 1$, and we do not worry about naming Alice's vertices.

We can compare Theorem 4.2.3 to the result of Gasarch and Hirst [13] that $\mathsf{WKL_0}$ is equivalent to the statement that every locally r-colorable (possibly infinite) graph is r-colorable, and observe that a sequence of finite graphs can be viewed as an infinite graph.

Theorem 4.2.4 ($\mathsf{RCA_0}$)**.** *Let $\mathcal{C}$ be a universal class of graphs, and let $r \in \mathbb{N}$.*

- *If $\mathsf{Col}(\mathcal{C}, r)$ is on-line solvable, then $\mathsf{SeqCol}(\mathcal{C}, r)$ holds.*
- *For any $k \in \mathbb{N}$, suppose that $\mathsf{Col}(\mathcal{C}_{\leq k}, r)$ is not on-line solvable and $\mathsf{Predict}_k(M)$ holds, where $M = \max\{2^k, r\}$. Then $\mathsf{SeqCol}(\mathcal{C}_{\leq k}, r)$ fails.*

Proof. Theorems 3.2.2 and 3.2.5.

□

Corollary 4.2.5 ($\mathsf{RCA_0}$)**.** *Suppose $\mathcal{C}$ is a k-bounded universal class of graphs and suppose $\mathsf{Predict}_k(r)$ holds. Then $\mathsf{SeqCol}(\mathcal{C}, r)$ holds if and only if Bob has a winning strategy in the game $\mathsf{G}(\mathcal{C}, r)$.*

Note that if $\mathcal{C}$ is a Π^0_1 universal class of graphs then there is a primitive recursive listing of finite graphs such that a graph G belongs to the class $\mathcal{C}$ if and only if no finite subgraph of G is isomorphic to a graph in this list.

Theorem 4.2.6. *Suppose there is k such that $\mathcal{C}$ is a k-bounded universal class of graphs. For every $r \geq 1$, the following are equivalent:*

- $\mathsf{Col}(\mathcal{C}, r)$ *is on-line solvable.*
- $\mathsf{RCA_0} \vdash \mathsf{SeqCol}(\mathcal{C}, r)$.

Proof. Suppose that $\mathsf{Col}(\mathcal{C}, r)$ is on-line solvable. Fix a bound k on the graphs in $\mathcal{C}$. Then k is standard, and the class $\mathcal{C}$ has a primitive recursive definition. Thus both $\mathcal{C}$ and k are expressible in $\mathsf{RCA_0}$, and Bob's winning strategy is a clear uniformly computable procedure that is definable in $\mathsf{RCA_0}$. So $\mathsf{RCA_0} \vdash \mathsf{Col}(\mathcal{C}, r)$ and Theorem 4.2.4 shows that $\mathsf{SeqCol}(\mathcal{C}, r)$ holds. For the other direction, assume that $\mathsf{Col}(\mathcal{C}, r)$ is not on-line solvable. Since $\mathsf{G}(\mathcal{C}, r)$ is determined (assuming all of second-order arithmetic), this means that Alice has a winning strategy in $\mathsf{G}(\mathcal{C}_{\leq k}, r)$ for some standard number k. By the second part of Theorem 4.2.4, since $\mathsf{SeqCol}(\mathcal{C}_{\leq k}, r)$ holds, we know that $\mathsf{Evade}_k(r)$ holds, implying $\mathsf{WKL_0}$ by Corollary 2.3.13. $\square$

Corollary 4.2.7. *Suppose $\mathcal{C}$ is a universal class of graphs with a primitive recursive code. For every r, $1 \leq r < \omega$, the following are equivalent:*

- *Bob has a winning strategy in* $\mathsf{G}(\mathcal{C}, r)$.
- *For every $k < \omega$,* $\mathsf{RCA_0} \vdash \mathsf{SeqCol}(\mathcal{C}_{\leq k}, r)$.

Corollary 4.2.8. *Let $\mathcal{C}$ be a universal class of graphs with a primitive recursive code which is not on-line colorable with any number of colors.*
Then $\mathsf{RCA_0} \vdash \forall r(\mathsf{SeqCol}(\mathcal{C}, r) \rightarrow \exists k \mathsf{Evade}_k(r))$.

Proof. This is a specific instance of Corollary 3.5.2.

□

4.3 Marriage Problems

This section is joint work with Dorais.

A *marriage problem* is a triple (B, G, R) where B, G are sets and $R \subseteq B \times G$. A *matching* is a one-to-one partial function $B \to G$ whose graph is contained in R; a *perfect matching* is one which is total. In anthropomorphic terms, B is a set of boys, G is a set of girls and $(b, g) \in R$ (also written $b\ R\ g$) indicates that boy b and girl g can see each other. A *matching* is a marriage between boys and girls who can see each other where polygamy and polyandry are forbidden.

In this section, we will always say that "boy b *sees* girl g" to indicate that $b\ R\ g$.

Hall's Theorem [18] gives a necessary and sufficient condition for a finite marriage problem to have a perfect matching, which is the case $\alpha = 1$ of the following definition.

Definition 4.3.1. Given a positive real number α, a marriage problem (B, G, R) satisfies the *α-Hall condition* if for every finite set $B_0 \subseteq B$, the set

$$RB_0 = \{g \in G : (\exists b \in B_0)(b\ R\ g)\}$$

has size at least $\alpha|B_0|$.

For positive real numbers α and β, $\mathsf{Match}(\alpha, \beta)$ will denote the statement that every *finite* marriage problem (B, G, R) that satisfies the α-Hall condition has a matching M of size at least $\beta|B|$. $\mathsf{Match}_k(\alpha, \beta)$ denotes the same statement where, in addition, one requires that $G = \{1, \ldots, k\}$. We make this precise by defining it using the language of Chapter 3:

Definition 4.3.2. Let α and β be positive real numbers. $\mathsf{Match}(\alpha, \beta)$ is the problem (A, C, S), where $A = \mathbb{N}^{<\infty}$, $C = \mathbb{N}^{<\infty}$, and $(\bar{a}, \bar{c}) \in S$ if the following is true: If the binary representation of a_i encodes the set of girls that boy i sees, and if the resulting graph satisfies the α-Hall condition, then if we match boy i to girl c_i if $c_i > 0$, and we do not match boy i if $c_i = 0$, our resulting function M is a valid matching of size at least $\beta|\bar{a}|$.

With this terminology, Hall's Theorem states that $\mathsf{Match}(1, 1)$ is true. Sequential versions of Hall's Theorem were investigated by Fujiwara and Yokoyama [12] who showed that an unbounded sequential version of $\mathsf{Match}(1, 1)$ is equivalent to ACA_0 over RCA_0. As a consequence of Theorems 4.3.6 and 4.3.7 below, we will see that $\forall k \mathsf{SeqMatch}_k(1, 1)$ is equivalent to WKL_0 over RCA_0. These last two results should be compared with similar results of Hirst and Hughes [23] who studied infinite marriage problems where each boy sees only finitely many girls; an infinite branch through A would encode such an infinite marriage problem.

Proposition 4.3.3 (RCA_0)**.** *For every $\alpha > 0$, $\mathsf{Match}(\alpha, \min(\alpha, 1))$ holds.*

Proof. The König Duality Theorem says that every bipartite graph has a matching M and a vertex cover C such that every edge in M has exactly one vertex in C. For finite bipartite graphs, the proof can be carried out in RCA_0. It follows that for every finite marriage problem (B, G, R) there is a matching M and sets $B_0 \subseteq B$, $G_0 \subseteq G$ such that each pair in R contains either a boy from B_0 or a girl from G_0, but no boy from B_0 is matched with a girl from G_0 by M. It follows from this that $R(B - B_0) \subseteq G_0$ and $|M| = |B_0| + |G_0|$. If (B, G, R) satisfies the α-Hall condition, then $|G_0| \geq \alpha(|B| - |B_0|)$ and hence

$$|M| = |B_0| + |G_0| \geq \alpha|B| + (1 - \alpha)|B_0| \geq \min(\alpha, 1)|B|,$$

as required. $\square$

Proposition 4.3.4 (RCA_0). *For all $\alpha > 0$, $\mathsf{Match}(\alpha, \alpha/(1+\alpha))$ is on-line solvable. By Theorem 3.2.2, this implies $\mathsf{SeqMatch}(\alpha, \alpha/(1+\alpha))$.*

Proof. The matching produced by applying the greedy algorithm (or any on-line algorithm that systematically produces a maximal matching) to each marriage problem has this property. Indeed, if (B, G, R) satisfies the α-Hall condition and M is a maximal matching and B_0 is the set of boys that aren't matched in M, then RB_0 must be contained in the set of girls that are matched in M by the maximality of M. Thus $|M| = |B| - |B_0| \geq |RB_0| \geq \alpha|B_0|$, which means that $|B| \geq (1+\alpha)|B_0|$ and hence that $|M|/|B| \geq \alpha/(1+\alpha)$. $\square$

Theorem 4.3.5 (RCA_0). *Let $k \in \mathbb{N}$, and let α, β be positive real numbers. If $d, e \in \mathbb{N}$ are such that $k \geq \alpha(d+e)$ and $\alpha e \leq d < \beta(d+e)$, then $\mathsf{Match}_k(\alpha, \beta)$ is not on-line solvable.*

Proof. First some algebra reveals that

$$\frac{\alpha}{(1+\alpha)} \leq \frac{\frac{d}{e}}{1+\frac{d}{e}} = \frac{d}{d+e} < \beta$$

contrasting this with Proposition 4.3.4.

We build a marriage problem with k girls, which will have at most $d+e$ boys in the matching (but potentially more unmatched boys in B). In the two-player game, Alice will allow boys to see all k girls, and Bob will decide his own matching, until there are d boys matched. After the first d, however, the next e boys will see precisely the girls that the first d boys are matched with. (Note that if Bob waits indefinitely to make the d^{th} matching, he will lose.)

Any subset of the last e boys sees at least $d \geq \alpha e$ girls, satisfying the α-Hall condition. Any other subset of boys sees at least $k \geq \alpha(d+e)$ girls, also satisfying the α-Hall condition. However, only d boys are matched to a girl, and $d < (d+e)\beta$; therefore, our problem witnesses the failure of $\mathsf{Match}_k(\alpha, \beta)$.

□

By the results of Chapter 3, we can conclude:

Theorem 4.3.6. *Let α and β be positive rational numbers.*

- *If $\beta \leq \alpha/(1+\alpha)$, then $\mathsf{RCA_0} \vdash \forall k \mathsf{SeqMatch}_k(\alpha, \beta)$.*
- *If $\alpha/(1+\alpha) < \beta \leq \min(1, \alpha)$, then $\mathsf{RCA_0} \vdash$ (if $\mathsf{Predict}_{d+e}(2^k n)$ holds for some d, e, k with $k \geq \alpha(d+e)$ and $\alpha e \leq d < \beta(d+e)$ then $\neg\mathsf{SeqMatch}_k(\alpha, \beta)$).*
- *If $\alpha/(1+\alpha) < \beta \leq \min(1, \alpha)$, then $\mathsf{RCA_0} \vdash \forall k \mathsf{SeqMatch}_k(\alpha, \beta) \to \mathsf{WKL_0}$.*

Proof. The first two statements follow from Theorem 3.2.2, Theorem 3.2.5, Proposition 4.3.4, and Theorem 4.3.5. For the third, choose d, e such that $\alpha/(1+\alpha) < d/(d+e) < \beta$, and choose $k \geq \alpha(d+e)$. Then $\mathsf{Predict}_{d+e}(k+1)$ fails, and since clearly we can choose these d, e, k to be standard, we have $\mathsf{WKL_0}$ by Corollary 2.3.13.

□

The restriction to rational numbers α, β is only so that the statements make sense.

We now show that the bounded problem $\mathsf{Match}_k(\alpha, \beta)$, has a solvable closed kernel, implying that $\mathsf{WKL_0}$ is sufficient as well as necessary for solving its sequential version:

Theorem 4.3.7. *Let $k \in \mathbb{N}$ and let α, β be positive real numbers, with $\beta \leq \min(\alpha, 1)$. Then the closed kernel of $\mathsf{Match}_k(\alpha, \beta)$ is solvable; that is, there exists a solution all of whose initial segments also constitute solutions to their restricted marriage problems.*

Proof. It suffices to show that this is true for $\beta = \min(\alpha, 1)$. Let us restrict our attention to $\mathsf{Match}_k(\alpha, \alpha)$ with $\alpha \leq 1$.

Let $\bar{a}$ be a request from Alice, $\bar{a} = \langle a_0, \ldots, a_n \rangle$. Let C_1 be the set of all possible responses $\bar{c}$ from Bob such that $\bar{a}\ S\ \bar{c}$; such a set C_1 exists and is finite since entries of $\bar{c}$ are in $\{0, 1, \ldots, k\}$. For each $\bar{c} \in C_1$, define a new vector $\overline{z(c)}$ given by $z(c_i) = |j \leq i : c_j = 0|$. In other words, $\overline{z(c)}$ is an increasing vector such that $z(c_i)$ gives the number of unmatched boys up to boy i. Order the vectors $\overline{z(c)}$, $c \in C_1$ in reverse lexicographic order; so that $\overline{z(c)} \prec \overline{z(c')}$ if and only if $(\exists i)\,(z(c_i) < z(c'_i) \wedge (\forall j > i)\, z(c_j) = z(c'_j))$. This gives highest priority to responses that match the most boys; among these, highest priority is to responses that match the most boys before the last boy, and so on. Let $\bar{d}$ be a (not necessarily unique) response such that $(\forall \bar{c} \in C_1)\ \overline{z(d)} \preceq \overline{z(c)}$.

I claim that $\bar{d}$ is a solution to the closed kernel of $\mathsf{Match}_k(\alpha, \alpha)$. Let M be the matching encoded by $\bar{d}$, and recall that $\bar{a}$ encodes the relation (B, G, R) with a_i encoding the girls that boy i sees. Let $i < n$, and suppose that $\langle a_0, \ldots, a_i \rangle\ S\ \langle d_0, \ldots, d_i \rangle$ fails, so that if $B_i = \{0, \ldots, , i\}$, $|RB_i| < \alpha(i+1)$. Since (B, G, R) satisfies the α-Hall condition, there does exist a matching L_i of size at least $\alpha(i+1)$ for the set B_i. Modify the original matching M to a new matching L as follows: (i)

(i) If $(j, g) \in L_i$ and $(j', g) \in M$, then replace (j', g) with (j, g) in L; (ii) If $(j, g) \in L_i$ and j is not matched by M and also does not satisfy (i), then add (j, g) to L. (iii) All other pairs in M remain in L. Then L is a valid matching with $|L| \geq |M| \geq \alpha(n+1)$. However, if $\bar{\ell}$ is the vector of responses associated to L, then $\overline{z(\ell)} \prec \overline{z(d)}$, since $z(\ell_i) < z(d_i)$ and $z(\ell_j) \leq z(d_j)$ for $i \leq j \leq n$. This contradicts the choice of $\bar{d}$. Therefore, $\bar{d}$ is a solution to the closed kernel of $\mathsf{Match}_k(\alpha, \alpha)$.

□

Corollary 4.3.8 (RCA_0). *If* $\alpha/(1+\alpha) < \beta \leq \min(1, \alpha)$ *and* $k < \omega$, *then* $\mathsf{SeqMatch}_k(\alpha, \beta) \leftrightarrow \mathsf{WKL}_0$.

Fujiwara and Yokoyama [12] showed that an unbounded sequential version of $\mathsf{Match}(1,1)$ is equivalent to $\mathsf{ACA_0}$ over $\mathsf{RCA_0}$. We show that the same is true of the unbounded non-optimal case:

Theorem 4.3.9 ($\mathsf{RCA_0}$)**.** *Let α, β be positive rational numbers, and let $\alpha/(1+\alpha) < \beta \leq \min(1,\alpha)$. Then the following are equivalent:*

(i) $\mathsf{ACA_0}$

(ii) The sequential unbounded matching problem with parameters α, β: Given a sequence $\langle B_n, G_n, R_n\rangle_{n\in\mathbb{N}}$, where B_n, G_n are finite sets and $R_n \subseteq B_n \times G_n$, and such that for all $n \in \mathbb{N}$, $\langle B_n, G_n, R_n\rangle$ satisfies the α-Hall condition, there exists a sequence of matchings $\langle M_n\rangle_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, M_n is a matching for $\langle B_n, G_n, R_n\rangle$ and $|M_n| \geq \beta|B_n|$.

Proof. (i) $\Rightarrow$ (ii) by the proof of Proposition 2.1.4.

(ii) $\Rightarrow$ (i): Assume (ii). The witnessing example will be very similar to the one in Theorem 4.3.5. As in that example, we choose d, e such that $\alpha/(1+\alpha) < \dfrac{d}{d+e} < \beta$, which also implies that $\alpha e < d$. Choose any $k \geq \alpha(d+e)$.

We define a sequence of finite relations $\langle B_n, G_n, R_n\rangle_{n\in\mathbb{N}}$. Let $f : \mathbb{N} \to \mathbb{N}$ be an arbitrary injection; our goal is to prove the existence of the range of f, which will imply $\mathsf{ACA_0}$.

Define R_n as follows: $(i,j) \in R_n$ for all $0 \leq i,j < d$. If $f(s) = n$, then $(i, d+(k-d)s), (i, d+(k-d)s+1), \ldots, (i, d+(k-d)s+(k-d)) \in R_n$ for all $i < d$, and also $(d+s, j), (d+1+s, j), \ldots, (d+e-1+s, j) \in R_n$ for all $j < d$.

Let $\langle M_n\rangle_{n\in\mathbb{N}}$ be the matching guaranteed by (ii). By the same argument as in Theorem 4.3.5, R_n satisfies the α-Hall condition. However, if $n \in \operatorname{ran}(f)$, then M_n must match at least one of the first d boys to a girl higher than d. For the matching

3		
1		
1	3	
1	3	6

6		
3	3	3
1	1	1

Figure 4.1: The optimal off-line solution (left) and on-line solution (right)

must have size at least $(d+e)\beta > d$, which can only happen if one of the first d boys is matched to a girl higher than d. In conclusion,

$$n \in \operatorname{ran}(f) \;\leftrightarrow\; R_n(\{d, \ldots, d+e-1\}) \cap \{0, \ldots, d-1\} \neq \varnothing$$

which we can check computably. So $\operatorname{ran}(f)$ exists, and ACA_0 holds.

□

4.4 The Task Scheduling Problem

We now turn to the problem of scheduling a series of simultaneous tasks on k processors. Tasks of varying processing time are assigned to the scheduler, whose duty it is to choose a machine for each task. The scheduler's goal is to minimize the *completion time*, also called the *makespan*, which is the total time required to complete all N tasks. Of course, the scheduler has no knowledge of future tasks or their processing times.

Not surprisingly, the scheduler cannot always guarantee the optimal completion time. For example, if there are $k = 3$ processors, and the sequence of tasks have processing times (1, 1, 1, 3, 3, 3, 6), the optimal processing time is 6, but if the process stops before the final task, the optimal assignment is no longer optimal. (This example was already introduced in Section 1.2; see Figure 4.1.)

Given an on-line scheduling algorithm, we are concerned with the *competitive* *ratio*, between the algorithm's processing time and the optimal off-line processing time. In Figure 4.1, any on-line algorithm would have competitive ratio at least 5/3.

Definition 4.4.1. Suppose that we have k processors, and there is a sequence of at most N tasks to be completed. A function $\mu \subseteq: N \to [0, \infty)$ defines the processing time of each task. Suppose that $g \subseteq: N \to k$ assigns one of the k processors to each task, with $\operatorname{dom} g = \operatorname{dom} \mu$.

- We define the *completion time* of μ with respect to g, $\omega(\mu, g)$, as follows:

$$\omega(\mu, g) = \max_{i<k} \sum_{t \in g^{-1}(i)} \mu(t)$$

- We define the *completion time* of μ with respect to g after j tasks, $\omega_j(\mu, g)$:

$$\omega_j(\mu, g) = \max_{i<k} \sum_{t \in g^{-1}(i),\, t \leq j} \mu(t)$$

- We define a new function $z(\mu) \subseteq: N \to [0, \infty)$ as follows:

$$(z(\mu))(j) = \min\{\, \omega_j(\mu,\, \sigma) \mid \sigma : \{0, \ldots, j\} \to \{0, \ldots, k-1\}\,\}$$

- We define $\mathrm{OPT}(\mu) = \sup_j\, (z(\mu))(j)$.

- The *competitive ratio of* (μ, g), $\mathrm{CR}(\mu, g)$, is defined as:

$$\mathrm{CR}(\mu, g) \;=\; \sup_j \frac{\omega_j(\mu, g)}{(z(\mu))(j)}$$

Intuitively, $(z(\mu))(j)$ gives the lowest possible processing time of tasks $0, \ldots, j$, taken over all possible processor assignments of tasks 0 through j. Note that the vector $z(\mu)$ is uniformly computable from μ, but the quantities $\mathrm{OPT}(\mu)$ and $\mathrm{CR}(\mu, g)$ in general are not, since they require knowing the exact domain sizes.

If we have a well-defined algorithm A that takes a sequence of processing times and outputs a sequence of processors, then $A(\mu)$ is defined to be the completion time of μ when applying the algorithm A. The competitive ratio of A, $\mathrm{CR}(A)$, is defined to be the maximum ratio of A's processing time to the optimal processing time, taken over all possible processing times μ. More precisely:

Definition 4.4.2. Let A be a well-defined algorithm that takes as input a sequence of tasks μ and outputs a processor assignment $g_{A,\mu}$.

- We define

  $$A(\mu) = \omega(\mu, g_{A,\mu})$$

  If we let OPT be the optimal off-line algorithm, then this definition of $\mathrm{OPT}(\mu)$ coincides with the definition above.

- Let A be a well-defined algorithm. The *competitive ratio of* A, $\mathrm{CR}(A)$, is defined as:

  $$\mathrm{CR}(A) \;=\; \sup_{\mu \subseteq: N \to [0,\infty)} \frac{A(\mu)}{\mathrm{OPT}(\mu)} \;=\; \sup_{\mu \subseteq: N \to [0,\infty)} \mathrm{CR}(\mu, g_{A,\mu})$$

An algorithm's competitive ratio $\mathrm{CR}(A)$ is unlikely to be computable, as the definition quantifies over all possible sequences of real task times.

In the following definition, we should view M as a set of sequences of processing times μ, and we should view G as a set of sequences of processor assignments g.

Definition 4.4.3. $\mathsf{Sch}(N, k, \alpha)$ is the problem $\mathsf{P}(M, G, R)$, where $M = \mathbb{Q}^{<N}$, $G = k^{<N}$, and $\bar{\mu} \; R \; \bar{g}$ holds if and only if

$$\omega_r(\bar{\mu}, \bar{g}) \leq \alpha \cdot (z(\bar{\mu}))(r)$$

where $r = |\mu|$.

In other words, taking tasks given by μ and scheduling them according to g is *eventually* at most α times the optimal processing time.

$\mathsf{SeqSch}(N, k, \alpha)$ is defined in the usual way. It is easy to see that $\mathsf{Sch}(N, k, \alpha)$ is on-line solvable according to Definition 3.1.10 if and only if it is on-line solvable in the intuitive sense.

Remark 4.4.4. $\mathsf{Sch}(N, k, \alpha)$ implicitly argues the existence of an optimal algorithm, but it cannot explicitly compute the optimal algorithm. This is why $z(\mu)$ is used in Definition 4.4.3 rather than any reference to an actual optimal algorithm, including the quantities $\mathrm{OPT}(\mu)$ and $\mathrm{CR}(\mu, g)$. When we consider $\mathsf{SeqSch}(N, k, \alpha)$, the asserted sequence of schedulings $\langle \bar{g}_n \rangle_{n \in \mathbb{N}}$ cannot explicitly compute the sequence of optimal schedulings, or even the sequence of cardinalities of $\langle \bar{g}_n \rangle_{n \in \mathbb{N}}$. Had Definition 4.4.3 referred to $\mathrm{OPT}(\mu)$ rather than $z(\mu)$, this would not be the case, drastically altering the reverse-mathematical strength of $\mathsf{SeqSch}(N, k, \alpha)$.

The most naive on-line algorithm is to schedule each task on the machine with the current lightest load. This algorithm is often called LOW in the literature [27]. Graham [14] showed that this algorithm has competitive ratio $2 - \frac{1}{k}$:

Theorem 4.4.5 (Graham)**.** $\mathrm{CR}(\mathrm{LOW}) = 2 - \frac{1}{k}$.

Corollary 4.4.6. *Let $N, k < \omega$.*

- $\mathsf{RCA}_0 \vdash \mathsf{SeqSch}(N, k, 2 - \frac{1}{k})$.
- $\mathsf{RCA}_0 \vdash (\forall N)(\forall k)\, \mathsf{SeqSch}(N, k, 2)$.

Proof. Since the on-line LOW algorithm (which is easily formalized in second-order arithmetic) witnesses that $\mathsf{Sch}(N, k, 2 - \frac{1}{k})$ is on-line solvable, the corollary follows from Theorems 3.2.2 and 4.4.5.

□

The example in Figure 1 showed that no possible on-line algorithm can have a competitive ratio of 5/3. Another better, though not optimal, example is shown in Faigle et al. [10].

Proposition 4.4.7 (RCA_0)**.** *Let A be any on-line scheduling algorithm, and let* OPT *be the optimal off-line scheduling algorithm. Given the sequence of processing times $\mu = (1, 1, \ldots, 1, 1+\sqrt{2}, 1+\sqrt{2}, \ldots, 1+\sqrt{2}, 2+2\sqrt{2})$, with d iterations of both* 1 *and $1+\sqrt{2}$, we have*

$$A(\mu) \geq (1 + \frac{1}{\sqrt{2}}) \times \mathrm{OPT}(\mu).$$

Proof. In Figure 4.2, each column represents a processor, and the numbers in the column represent processing times of tasks assigned to that processor. It is easy to check that the top table is optimal, and that the bottom table follows the best possible on-line algorithm. The competitive ratio is

$$\frac{4+3\sqrt{2}}{2+2\sqrt{2}} = 1 + \frac{1}{\sqrt{2}}.$$

As the sequence is finite and all cases can be checked easily, RCA_0 is sufficient to prove this.

□

Note that $1 + \frac{1}{\sqrt{2}} \approx 1.707$. Albers [1] has improved the lower bound to 1.852 for $k = \operatorname{ran}\mu$ a multiple of 40. She has also improved the upper bound by creating an on-line algorithm with competitive ratio 1.923.

Theorem 4.4.8 (Albers)**.**

		$1+\sqrt{2}$			
	$1+\sqrt{2}$	1	$1+\sqrt{2}$	$\cdots$	$1+\sqrt{2}$
$2+2\sqrt{2}$	$1+\sqrt{2}$	1	1		1

$2+2\sqrt{2}$				
$1+\sqrt{2}$	$1+\sqrt{2}$	$1+\sqrt{2}$	$\cdots$	$1+\sqrt{2}$
1	1	1		1

Figure 4.2: The optimal off-line solution (top) and on-line solution (bottom)

- *There exists an on-line algorithm A such that* $\mathrm{CR}(A) = 1.923$.
- *Let k be a multiple of 40. There exists a sequence of tasks $\mu \subseteq: 4k+1 \to \mathbb{Q}^+$ such that for any task assignment $g : 4k+1 \to k$, we have* $\mathrm{CR}(\mu, g) \geq 1.852$.

So the best competitive ratio for an on-line scheduling algorithm is a currently unknown $\alpha \in (1.852, 1.923]$. In the special case of four machines, Rudin [31] has improved the lower bound to 1.88. (These decimals are exact.)

Theorem 4.4.9 ($\mathsf{RCA_0}$)**.** *Let $\alpha \geq 1.923$, $\alpha \in \mathbb{Q}$, and let $n, k \in \mathbb{N}$. Then* $\mathsf{SeqSch}(N, k, \alpha)$ *holds.*

Proof. This follows from Theorems 3.2.2 and 4.4.8. The latter has to be modified to be definable in second-order arithmetic: There exists an algorithm A such that for any $N, k \in \mathbb{N}$, A will take as input a sequence of processing times $\mu \subseteq: N \to \mathbb{Q}^+$ and will output a sequence of processors $g_{A,\mu} \subseteq: N \to k$ as described in Definition 4.4.2. For any such μ, we have $\mathrm{CR}(\mu, g_{A,\mu}) \leq 1.923$. Albers's proof in [1] is multi-step but straightforward, and can be carried out using Π^0_1-induction.

□

Theorem 4.4.10 ($\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$)**.** *Let* $\alpha < 1 + \frac{1}{\sqrt{2}}$*,* $\alpha \in \mathbb{Q}$*,* $k \in \mathbb{N}$*. Then* $\mathsf{SeqSch}(2k + 1, k, \alpha) \to \mathsf{ACA}_0$*.*

Proof. We will show that the example in Proposition 4.4.7, illustrated in Figure 4.2, does not have a solvable closed kernel; the statement then follows from Proposition 4.4.7 and Theorem 3.3.14. It is clear that the optimal solution does not have a closed kernel (at the stage when only the 1's are entered, the competitive ratio of the initial segment is 2). We have to be careful, though; there are other solutions that are not optimal, but do have competitive ratio $< 1 + \frac{1}{\sqrt{2}} \approx 1.707$.

First, note that any solution that assigns any two of the 1's to the same processor has an initial segment that is not a solution, for the exact reason stated above. The other two sensible possibilities are illustrated in Figure 4.3. The top table in the figure has a processing time of $4 + 3\sqrt{2}$, just like the non-solution in Figure 4.2 with that processing time, and is therefore also not a solution. The bottom table in the figure has a processing time of $3 + 2\sqrt{2}$ and therefore has competitive ratio

$$\frac{3 + 2\sqrt{2}}{2 + 2\sqrt{2}} = \frac{1}{2} + \frac{1}{\sqrt{2}} \approx 1.207$$

So it is indeed a solution. However, at the second-last stage, the optimal processing time is $2 + \sqrt{2}$, giving a competitive ratio of

$$\frac{3 + 2\sqrt{2}}{2 + \sqrt{2}} = 1 + \frac{1}{\sqrt{2}} \approx 1.707$$

Therefore, there are one optimal solution and one non-optimal solution (modulo permuting the processors with one another), both of which fail to be solutions at some initial segment. So $\mathsf{Sch}(2k + 1, k, \alpha)$ does not have a solvable closed kernel, and so $\mathsf{SeqSch}(2k + 1, k, \alpha) \to \mathsf{ACA}_0$.

□

	$1+\sqrt{2}$				
	$1+\sqrt{2}$				
$2+2\sqrt{2}$	$1+\sqrt{2}$		$1+\sqrt{2}$	$\cdots$	$1+\sqrt{2}$
1	1	1	1		1

	$1+\sqrt{2}$			
$2+2\sqrt{2}$	$1+\sqrt{2}$	$1+\sqrt{2}$	$\cdots$	$1+\sqrt{2}$
1	1	1		1

Figure 4.3: Two other possible processor assignments. The top one is not a solution; the bottom one is a non-optimal solution.

Observe that the two solutions fail at different initial segments. If we let $\bar{a}$ be the sequence of processing times, $\bar{b_1}$ the assignment in the optimal solution, and $\bar{b_2}$ the assignment in the non-optimal solution in Figure 4.3, notice that $(\bar{a} \upharpoonright 4)\ R\ (\bar{b_1} \upharpoonright 4)$ holds but $(\bar{a} \upharpoonright 8)\ R\ (\bar{b_1} \upharpoonright 8)$ fails; by contrast, $(\bar{a} \upharpoonright 4)\ R\ (\bar{b_2} \upharpoonright 4)$ fails but $(\bar{a} \upharpoonright 8)\ R\ (\bar{b_2} \upharpoonright 8)$ holds.

Can the sequential problem be equivalent to $\mathsf{WKL_0}$ in any case? Or is there a dividing line for α that directly separates $\mathsf{RCA_0}$ from $\mathsf{ACA_0}$? The equivalent question would be if Alice can always win by playing *one particular* sequence of numbers, or if she simply has a strategy that depends on Bob's placements, as in the graph coloring problem?

Condition 4.4.11. *Let $\alpha \geq 1$, $N, k \in \mathbb{N}$. $\mathsf{UniformWin}(N, k, \alpha)$ is the condition that Alice has a particular winning play $\bar{a}$ that will defeat any processor assignment $\bar{b}$ from Bob at some initial segment of the problem; that is, $(\bar{a} \upharpoonright j)\ R\ (\bar{b} \upharpoonright j)$ fails for some $j < N$. In other words, $\bar{a}$ is a winning play for Alice in the closed kernel of $\mathsf{Sch}(N, k, \alpha)$.*

Conjecture 4.4.12 ($\mathsf{RCA_0}$). *Let $\alpha \geq 1$, $N, k \in \mathbb{N}$. Suppose that $\mathsf{Sch}(N, k, \alpha)$ is not on-line solvable. Then $\mathsf{UniformWin}(N, k, \alpha)$ holds. Moreover, the processing times in Alice's winning play are in nondecreasing order: $a_0 \leq a_1 \leq \cdots \leq a_{N-1}$.*

The example in Proposition 4.4.10 clearly witnesses the conjecture for $\alpha = 1 + \frac{1}{\sqrt{2}}$, and the conjecture is true in many lower-bound counterexamples in the literature, such as those in [1] (see Theorem 4.1) and [31] (see the table in Example 7). It seems nontrivial, for changing consecutive processing times from say $(20, 10)$ to $(10, 20)$ can open up new moves for Bob, if a placement option for the 20 becomes possible when it no longer exceeds α times the optimum, which can change when the 10 is played.

Proposition 4.4.13 ($\mathsf{RCA_0}$). *Let $\alpha \geq 1$, $\alpha \in \mathbb{Q}$, $N, k \in \mathbb{N}$ be parameters satisfying $\mathsf{UniformWin}(N, k, \alpha)$. Then the closed kernel of $\mathsf{Sch}(N, k, \alpha)$ is not solvable.*

Proof. Let $\bar{a}$ be Alice's uniformly winning play in the closed kernel of $\mathsf{Sch}(N, k, \alpha)$ from Condition 4.4.11, and let $\bar{b}$ be a response from Bob that is a winning play in $\mathsf{Sch}(N, k, \alpha)$ (not in its closed kernel). Since $\bar{a}$ will defeat all possible processor assignments including Bob's $\bar{b}$, it must be that $(\bar{a} \restriction j)\ R\ (\bar{b} \restriction j)$ for some $j < N$.

□

Theorem 4.4.14 ($\mathsf{RCA_0} + \mathsf{I}\Sigma^0_2$). *Let $\alpha \geq 1$, $\alpha \in \mathbb{Q}$, $N, k \in \mathbb{N}$. Suppose that $\mathsf{UniformWin}(N, k, \alpha)$ holds. Then $\mathsf{SeqSch}(N, k, \alpha) \leftrightarrow \mathsf{ACA_0}$.*

Proof. This follows from Theorem 3.3.14 and Proposition 4.4.13.

□

Corollary 4.4.15 ($\mathsf{RCA_0} + \mathsf{I}\Sigma^0_2$). *Let k be a positive multiple of 40 and let $\alpha \leq 1.852$. Then $\mathsf{SeqSch}(4k + 1, k, \alpha) \leftrightarrow \mathsf{ACA_0}$.*

Proof. This follows from Theorem 4.4.8 and Theorem 4.4.14; the former depends on Albers's work in [1] which is straightforward and provable in $\mathsf{RCA_0}$.

□

4.5 The Paging Problem

In this problem, we work with a memory system with two levels: a fast memory and a slow memory, with the fast memory having a fixed size k. Pages in the memory system will be requested, and our goal is to optimize the process so that, as often as possible, the requests are already in the fast memory.

We follow the terminology in [3]. The memory system receives a sequence of up to N page requests. If the requested page is in the fast memory, no cost incurs. If the requested page is in the slow memory, we move the requested page into the fast memory, and we move another page from the fast memory into the slow memory. In this case we have a "page fault" with cost 1.

We will let $\mu : N \to \mathbb{N}$ denote the sequence of page requests, and we will let $g : N \to k$ denote the sequence of assigned fast memory cells, so that for $j < N$, page $\mu(j) \in \mathbb{N}$ gets assigned fast memory cell $g(j) < k$.

The following definition formalizes some intuitive concepts from this problem.

Definition 4.5.1.

- Let $\mu \subseteq: N \to \mathbb{N}$, and let $g \subseteq N \to k$, with $\operatorname{dom} \mu = \operatorname{dom} g$. At the time of a request $\mu(i)$, we say that page m is the *last occupant of cell j in the fast memory* if for the greatest $i' < i$ such that $g(i') = j$, we have $\mu(i') = m$.

- If m is the last occupant of some cell j, we say that m is *in the fast memory at the time of request $\mu(i)$.*

- We say that g is a *proper assignment* if

– If $\mu(i) = m$ with m the last occupant of cell j in the fast memory, then $g(i) = j$ as well. (In particular, g never assigns a page m to two distinct cells in the fast memory at any particular time.)

– If there are cells in the fast memory that are empty at the time of request $\mu(i)$, then $g(i)$ will choose one of these cells.

If $\mu(i) = m$ for some m not already in the fast memory, and if the fast memory is full at time i, then we incur a page fault with cost 1.

Definition 4.5.2. The *cost* $C(\mu, g)$ is defined as

$$C(\mu, g) = \left| i < N \,:\, \text{all cells } j < k \text{ are occupied by pages other than } \mu(i) \text{ at time } i \right|$$

We can also define

$$C_i(\mu, g) = \left| i' < i \,:\, \text{all cells } j < k \text{ are occupied by pages other than } \mu(i') \text{ at time } i' \right|$$

Note that $C(\mu, g)$ is uniformly computable from μ and g.

Definition 4.5.3. If A is a well-definied algorithm that takes as input a sequence of page requests μ and outputs a sequence of assigned memory cells $g_{A,\mu}$, then we define

$$A(\mu) = C(\mu, g_{A,\mu})$$

.

Just as in Section 4.4, we create a new function $z(\mu)$ that essentially means "optimal cost so far"; see Remark 4.4.4 for the necessity of this concept.

Definition 4.5.4. We define $z(\mu) \subseteq: N \to \mathbb{N}$ as follows:

$$(z(\mu))(j) = \min\{\, C(\mu \upharpoonright j, \sigma) \mid \sigma : \{0, \ldots, j\} \to \{0, \ldots, k-1\} \,\}$$

Sleator and Tarjan [37] observe that the optimal off-line algorithm OPT uses the "Longest Forward Distance" algorithm: Always displace the page whose next access is latest. This is clearly the unique optimal algorithm.

The best on-line algorithm is LRU (Least Recently Used), where we always displace the least recently used page in the fast memory. Sleator and Tarjan show that LRU has a competitive ratio of k.

Theorem 4.5.5 (Sleator-Tarjan)**.** *Let μ be any request function, $\mu \subseteq N \to \mathbb{N}$. Then* $\mathrm{LRU}(\mu) \ \leq \ k \times \mathrm{OPT}(\mu) + k$.

Proposition 4.5.6 (Sleator-Tarjan)**.** *Let A be any on-line algorithm. Then there are arbitrarily long request functions μ such that $A(\mu) \geq k \times \mathrm{OPT}(\mu)$.*

Proof. First request k pages to fill up the fast memory, which costs nothing. Then request 1 page not in the fast memory. Then, for the next $k-1$ requests, request the exact page that you just displaced from the fast memory. Repeat this procedure as often as you want, say d times. A will fault every single time, kd times. However, OPT will fault at most d times, since when it does fault, it knows that at least one of the k pages in fast memory will not be requested in the next $k-1$ requests, so it will know to displace that page. So $A(\mu) = kd \ \geq \ k \times \mathrm{OPT}(\mu)$.

Here is the formal definition of a request sequence $\mu : N \to \mathbb{N}$ that will work:

- For $0 \leq i \leq k-1$, $\mu(i) = i$.

- For $i \geq k$, $i = kt$, we define $\mu(i) = kt$.

- For $i \geq k$, $i = kt + r$, $r \neq 0$, we define $\mu(i) = m$, where m is the last occupant of cell j of the fast memory, where $g_{A,\mu}(i-1) = j$.

$\square$

Definition 4.5.7. Let $N, k \in \mathbb{N}$, $\alpha \in \mathbb{Q}$. $\mathsf{Page}(N, k, \alpha)$ is the problem (M, G, R), where $M = \mathbb{N}^{<N}$, $G = k^{<N}$, and $\bar{\mu}\ R\ \bar{g}$ holds if and only if $\bar{g}$ is a proper assignment and if

$$C_r(\bar{\mu}, \bar{g}) \leq \alpha \cdot (z(\bar{\mu}))(r) + k$$

where $r = |\mu|$.

$\mathsf{SeqPage}(N, k, \alpha)$ is defined in the usual way.

Proposition 4.5.8. *Let $N, k \in \mathbb{N}$, $\alpha \geq k$. Then* $\mathsf{RCA}_0 \vdash \mathsf{SeqPage}(N, k, \alpha)$.

Proof. By Theorem 4.5.5, the LRU on-line algorithm has the property that $C(\mu, g_{\mathrm{LRU},\mu}) \leq k * \mathrm{OPT}(\mu) + k$. Since the LRU algorithm is easily definable in RCA_0, Theorem 3.2.2 implies that RCA_0 proves the sequential version. $\square$

Proposition 4.5.9 (RCA_0)**.** *Let $N, k \in \mathbb{N}$, $\alpha \geq 1$. Then* $\mathsf{Page}(N, k, \alpha)$ *has a solvable closed kernel.*

Proof. $\mathsf{Page}(N, k, 1)$ is clearly solvable by taking $\bar{g}$ to be the optimal assignment, and following our definition of $\mathsf{Page}(N, k, \alpha)$, every initial segment of the optimal solution is a solution.

$\square$

Corollary 4.5.10 (RCA_0)**.** *Let $N, k \in \mathbb{N}, \alpha \geq 1$. Then* $\mathsf{WKL}_0 \vdash \mathsf{SeqPage}(N, k, \alpha)$.

Proof. The problem is semi-bounded; apply Theorem 3.3.2 and Proposition 4.5.9.

$\square$

Proposition 4.5.11 (RCA_0)**.** *Let $d, k \in \mathbb{N}$, $1 \leq \alpha < k$. Then $\mathsf{Page}(dk, k, \alpha)$ is not on-line solvable.*

Proof. This follows from Proposition 4.5.6.

□

Corollary 4.5.12 (RCA_0)**.** *Let $d, k \in \mathbb{N}$, $1 \leq \alpha < k$, $dk < \omega$. Then $\mathsf{SeqPage}(dk, k, \alpha) \vdash \mathsf{WKL}_0$.*

Proof. This follows from Corollary 3.2.6 and Proposition 4.5.11.

□

4.6 The List Update Problem

In the List Update problem, we maintain a permutation σ of the numbers 1 through k. We receive a series of requests for numbers in $\{1, \ldots, k\}$. The cost of the request is the index of that number in the permutation σ, i.e., the distance from the left you have to walk to get to that number. Once the number is requested, we can assign it any space in the permutation that we wish, keeping the other numbers in the same order.

The optimal off-line algorithm, which we will call OPT, is to initially order the numbers in the order they will first be requested, and then once a number is requested, place it after all other numbers that will be requested before it. OPT has cost N, the total number of requests, since the requested number will always be at the beginning of the permutation.

For any on-line algorithm, there is a worst-case scenario where the algorithm incurs the maximum possible cost of Nk. Namely, whichever number is last in the permutation is always the next number requested. So if $\alpha < 1$, no on-line algorithm can guarantee a cost as low as $Nk\alpha$.

We will let $\mu \subseteq: N \to k$ denote a sequence of up to N fetch requests, and we will let $g : \{-k, \ldots, -1\} \cup (\operatorname{dom} \mu) \to k$ denote the sequence of assigned spaces. So $\mu(i)$ denotes the number requested in step i, and $g(i)$ denotes the space assigned to $\mu(i)$, for $i \geq 0$. $g(-j)$ denotes the initial space of the number $j - 1$, $1 \leq j \leq k$.

We can easily keep track of the full permutation after each request: We will define a sequence of permutations $\sigma : (\{-1\} \cup \operatorname{dom} \mu) \times k \to k$, so that $\sigma(i, j)$ represents the number in space j after the request $\mu(i)$, and $\sigma(-1, \_\_)$ is the initial permuation. At the same time we will define a cost $c : \operatorname{dom} \mu \to k$, such that the number $\mu(i)$ is in space $c(i)$ before it is requested. Both σ and c will be uniformly computable from μ and g, and will be defined recursively by:

- $\sigma(-1,\, j-1) = g(-j)$; (We have an initial permutation)
- $c(i)$ is the previously assigned space of the request $\mu(i)$, so that $\sigma(i-1,\, c(i)) = \mu(i)$. (The cost of the request is how far you have to walk).
- $\sigma(i, g(i)) = \mu(i)$; (Move the request to its assigned space.)
- $\sigma(i, j) = \sigma(i-1, j-1)$ if $g(i) < j \leq c(i)$ (If the request is moved left, intermediate elements are moved right).
- $\sigma(i, j) = \sigma(i-1, j+1)$ if $c(i) \leq j < g(i)$ (If the request is moved right, intermediate elements are moved left).
- $\sigma(i, j) = \sigma(i-1, j)$ for all other j.

Note that $c(i)$ is defined in terms of $\sigma(i-1, \_\_)$, and that $\sigma(i, \_\_)$ is defined in terms of $c(i)$.

Definition 4.6.1. Let $N, k \in \mathbb{N}, \alpha \in \mathbb{Q}$. $\mathsf{LU}(N, k, \alpha)$ is the problem (M, G, R), where $M = G = k^{<N}$ and $\bar{\mu}\ R\ \bar{g}$ holds if and only if

$$\sum_{i<|\mu|} c(i) \leq \alpha \cdot |\mu|$$

$\mathsf{SeqLU}(N, k, \alpha)$ is defined in the usual way. The extra function z from the previous two sections is not necessary here, since we explicitly know the cost of OPT.

Proposition 4.6.2. *Let $N, k < \omega$, $\alpha \geq k$. Then $\mathsf{RCA}_0 \vdash \mathsf{SeqLU}(N, k, \alpha)$.*

Proof. This is clear, since $\alpha = k$ is the worst-case scenario; see the explanation in the first paragraphs of this section.

□

Proposition 4.6.3 (RCA_0)**.** *Let $N, k \in \mathbb{N}$, $\alpha \geq 1$. Then $\mathsf{LU}(N, k, \alpha)$ has a solvable closed kernel.*

Proof. $\mathsf{LU}(N, k, 1)$ is clearly solvable by taking $\bar{g}$ to be the optimal assignment, and following our definition of $\mathsf{LU}(N, k, \alpha)$, every initial segment of the optimal solution is a solution.

□

Corollary 4.6.4 (RCA_0)**.** *Let $N, k \in \mathbb{N}$, $\alpha \geq 1$. Then $\mathsf{WKL}_0 \vdash \mathsf{SeqLU}(N, k, \alpha)$.*

Proof. The problem is bounded; apply Theorem 3.3.2 and Proposition 4.6.3.

□

Proposition 4.6.5 (RCA_0)**.** *Let $N, k \in \mathbb{N}$, $1 \leq \alpha < k$. Then $\mathsf{LU}(N, k, \alpha)$ is not on-line solvable.*

Proof. See the explanation in the opening paragraphs of this section.

□

Corollary 4.6.6. *Let* $N, k < \omega$, $1 \leq \alpha < k$. *Then* $\mathsf{SeqLU}(N, k, \alpha) \vdash \mathsf{WKL_0}$.

Proof. This follows from Corollary 3.2.6 and Proposition 4.6.5.

□

Essentially, on-line list update algorithms can never beat the worst case scenario. Sleator and Tarjan [37] analyze the naive "Move To Front" (MTF) algorithm that moves the requested number to the front. They show that, if we do not allow either algorithm to choose the initial permutation, then MTF has a competitive ratio of 2. (Example: It is relatively easy to see that in the worst case scenario when $N = k$, MTF incurs a cost of k^2, but OPT incurs a cost of $\binom{k}{2}$, an approximate ratio of 2.)

4.7 Dilworth's Theorems

A *partial order* $P = (V, \leq)$ is a reflexive, antisymmetric, transitive relation $\leq$ on the set V. We will refer to V as the set of vertices of P. A *chain* C is a subset $C \subseteq P$ that is totally ordered with respect to $\leq$, i.e., $(\forall c_1, c_2 \in C)\,[(c_1 \leq c_2) \vee (c_2 \leq c_1)]$. An *antichain* A is a subset $A \subseteq P$ such that no two elements of A are comparable, i.e.,

$(\forall a_1, a_2 \in A)\,[(a_1 \neq a_2) \rightarrow (\neg(a_1 \leq a_2) \wedge \neg(a_2 \leq a_1))]$.

If the largest antichain in P has size k, then we say the *width* of P is k. If the largest chain in P has size m, then we say the *height* of P is m.

Dilworth's Theorem states that if a partial order $P = (V, \leq)$ has width bounded by k, then there exists a set of k chains that cover P. Cenzer and Remmel [5] prove that Dilworth's Theorem is equivalent to $\mathsf{WKL_0}$ for infinite partial orders. Since a sequence of finite partial orders $\langle P_n \rangle_{n \in \mathbb{N}} = \langle V_n, \leq_n \rangle_{n \in \mathbb{N}}$ of width $\leq k$ can be combined into an infinite partial order of width $\leq k$ — just make the entire $V_n < V_{n'}$ whenever

$n < n'$ — it follows that $\mathsf{WKL_0}$ implies the sequential Dilworth's theorem. We will see that the two are equivalent.

Definition 4.7.1. Let $k \geq 2$. $\mathsf{Dilworth}(k)$ is the problem (A, B, R), where $A = \mathbb{N}^{<\infty}$, $B = k^{<\infty}$. $(\bar{a}\ R\ \bar{b})$ is true if the following holds: If $a_i < 3^i$, and *if* by using the ternary representation of a_i to determine comparability of vertex i with vertices $0, \ldots, i-1$, this results in a valid partial ordering $\leq$ with width at most k, *then* assigning vertices values in $\{0, 1, \ldots, k-1\}$ according to $\bar{b}$ results in k chains. That is, if $b_i = b_{i'}$, then $(i, i') \in \leq$ or $(i', i) \in \leq$.

$\mathsf{SeqDilworth}(k)$ is defined in the usual way.

Proposition 4.7.2. *Let $k \geq 2$. Then the closed kernel of* $\mathsf{Dilworth}(k)$ *is solvable.*

Proof. The finite problem is clearly solvable. Any initial segment of a solution clearly will encode a partial order with at most k chains as well. (The transitivity guarantees that if $v_1 < v_3 < v_2$, then the relation $v_1 < v_2$ already existed before v_3 was introduced.)

□

Corollary 4.7.3. *Let $k \geq 2$. Then* $\mathsf{WKL_0} \vdash \mathsf{SeqDilworth}(k)$.

Proof. The problem is semi-bounded. Apply Theorem 3.3.2 and Proposition 4.7.2.

□

The problem is, in fact, equivalent to a *bounded* problem, in which A consists of finite sequences $\bar{a}$ satisfying $a_i < 3^i$.

Theorem 4.7.4. *Let $k \geq 2$. Then* $\mathsf{Dilworth}(k)$ *is not on-line solvable.*

Proof. The partial order we will create is shown in Figure 4.4. In this order, we always have $0 < k$, $1 < k+1$, $2 < k+2, \ldots, k-1 < 2k-1$, as well as $0 < k+1$, $1 < k$. No

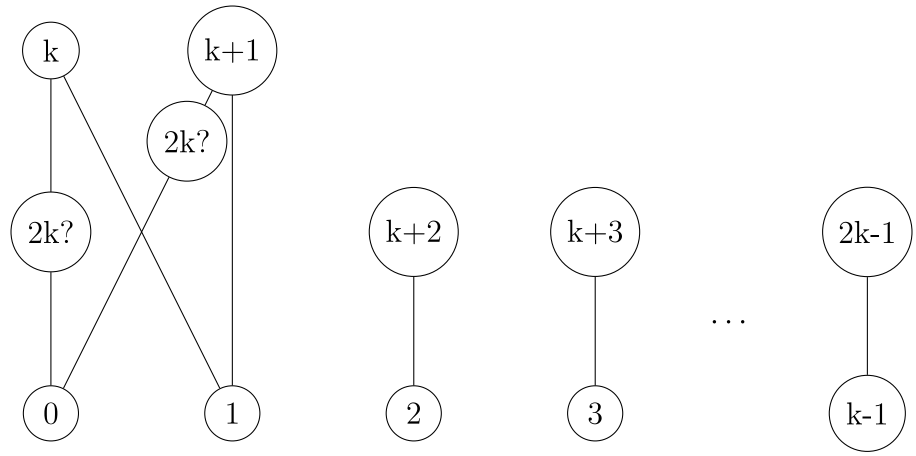

Figure 4.4: Partial order in the proof of Theorem 4.7.4. If there is a down-to-up path from v_1 to v_2 in this picture, then $v_1 < v_2$.

other relations hold among vertices 0, ..., $2k-1$. Recall that Alice will enumerate vertices in numerical order, and when she enumerates a vertex, she will list all order relations of that vertex with previous vertices. This means that Bob has very little freedom in assigning chains. In fact, he only has one choice: when k is enumerated, he can assign it to the same chain as 0 or as 1. Then $k+1$ will be assigned the other chain.

Alice can then define vertex $2k$ based on Bob's choice of chains. If Bob assigns 0, k to the same chain, Alice will choose $0 < 2k < k+1$, and Bob will not be able to correctly assign a chain to vertex $2k$. If Bob assigns 0, $k+1$ to the same chain, then Alice will choose $0 < 2k < k$, and Bob will similarly be defeated.

□

.

Corollary 4.7.5 (RCA_0). *Let* $2 \leq k < \omega$. *Then* $\mathsf{SeqDilworth}(k) \vdash \mathsf{WKL}_0$ *and* $\mathsf{SeqDilworth}(k) \leftrightarrow \mathsf{WKL}_0$.

Proof. This follows from Corollary 3.2.6 and Theorem 4.7.4.

□

The Dual to Dilworth's Theorem states that if a partial order $P = (V, \leq)$ has height bounded by m, then there exists a set of m antichains that cover P. Like Dilworth's Theorem, Cenzer and Remmel [5] prove that the dual theorem for infinite partial orders is equivalent to $\mathsf{WKL_0}$.

Definition 4.7.6. Let $m \geq 2$. $\mathsf{DualDilworth}(m)$ is the problem (A, B, R), where $A = \mathbb{N}^{<\infty}$, $B = m^{<\infty}$. $(\bar{a}\ R\ \bar{b})$ is true if the following holds: If $a_i < 3^i$, and *if* by using the ternary representation of a_i to determine comparability of vertex i with vertices $0, \ldots, i-1$, this results in a valid partial ordering $\leq$ with height at most m, *then* assigning vertices values in $\{0, 1, \ldots, m-1\}$ according to $\bar{b}$ results in m antichains. That is, if $b_i = b_{i'}$, then $(i, i') \notin \leq$ and $(i', i) \notin \leq$.

$\mathsf{SeqDualDilworth}(m)$ is defined in the usual way.

Proposition 4.7.7. *Let $m \geq 2$. Then the closed kernel of* $\mathsf{DualDilworth}(m)$ *is solvable.*

Proof. The finite problem is clearly solvable. Any initial segment of a solution clearly will encode a partial order with at most m antichains as well.

□

Corollary 4.7.8. *Let $m \geq 2$. Then* $\mathsf{WKL_0} \vdash \mathsf{SeqDualDilworth}(m)$.

Proof. The problem is semi-bounded (in fact, equivalent to a bounded problem). Apply Theorem 3.3.2 and Proposition 4.7.7.

□

Theorem 4.7.9. *Let $m \geq 2$. Then* $\mathsf{DualDilworth}(m)$ *is not on-line solvable.*

Proof. The partial order we will create is shown in Figure 4.5. In this order, we always have $0 < 2$, $0 < 3$, $1 < 3$, $1 < 4$, $2 < 3$, $2 < 4$, $3 < 5$, $3 < 6$, $4 < 5$, $4 <$

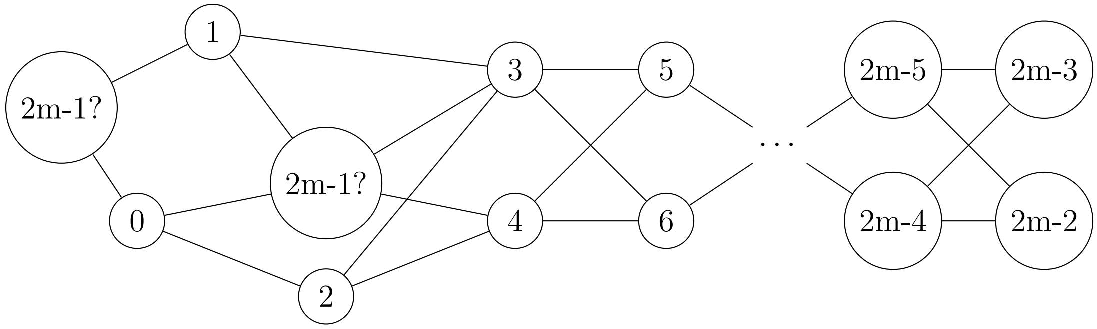

Figure 4.5: Partial order in the proof of Theorem 4.7.9. If there is a left-to-right path from v_1 to v_2 in this picture, then $v_1 < v_2$. In this order, $0 < 3$ and $1 < 4$ always.

6, $\ldots$, $2m-5 < 2m-3$, $2m-5 < 2m-2$, $2m-4 < 2m-3$, $2m-4 < 2m-2$, plus all transitive closures.

Recall that Alice will enumerate vertices in numerical order, and when she enumerates a vertex, she will list all order relations of that vertex with previous vertices. This means that Bob has very little freedom in assigning antichains. In fact, he only has one choice: when 1 is enumerated, he can assign it to the same antichain as 0 or to a different one. If he chooses the latter, he must choose 2 to belong to the same antichain as 1; otherwise his constraints will guarantee a width of $m+1$ and he will lose. So in short, Bob has two plays that will keep him in the game up through vertex $2m-2$: either 0 is in its own antichain and 1 and 2 share one, or 2 is in its own antichain and 0 and 1 share one.

Alice can then define vertex $2m-1$ based on Bob's choice of antichains. If Bob assigns 1, 2 to the same antichain, Alice will choose $2m-1 < 0$ and $2m-1 < 1$ plus the transitive closures, and Bob will not be able to correctly assign an antichain to vertex $2m-1$. If Bob assigns 0, 1 to the same antichain, then Alice will choose $0 < 2m-1 < 3$ and $1 < 2m-1 < 4$ plus the transitive closures, and Bob will similarly be defeated. Note that in either case, the height of the partial order is still m.

$\square$

.

Corollary 4.7.10 ($\mathsf{RCA_0}$)**.** *Let* $2 \leq m < \omega$. *Then* $\mathsf{SeqDualDilworth}(m) \vdash \mathsf{WKL_0}$ *and* $\mathsf{SeqDualDilworth}(m) \leftrightarrow \mathsf{WKL_0}$.

Proof. This follows from Corollary 3.2.6 and Theorem 4.7.9.

$\square$

Remark 4.7.11. Note that in both Theorem 4.7.4 and in Theorem 4.7.9, the proofs would be identical if we only had the comparability graphs rather than the full relations $\leq$. Also, both proofs would work if we assumed the partial orders had to be connected: in Theorem 4.7.9 the partial order we constructed *was* connected, and in Theorem 4.7.4 we could easily make the order connected by adding diagonal relations $1 < k+2$, $2 < k+3$, $\ldots$, $k-2 < 2k-1$, and this would not change the assignments of chains.

4.8 Ramsey's Theorem

Given a set $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $[A]^n$ denote the collection of subsets of A of size n.

Let $k \geq 2$ be a number of colors. Our version of the finite Ramsey's Theorem for n-tuples states that for every finite set of vertices V, there exists a finite superset of vertices $W \supseteq V$, such that for any coloring $g : [W]^n \to k$, there is a homogeneous set H of size $|V|$:

$$(|H| = |V|) \wedge (H \subseteq W) \wedge (\exists c < k)(\operatorname{ran} g \restriction [H]^n = \{c\})$$

Let us work for the moment with pairs ($n = 2$) and two colors ($k = 2$). Since various results in finite combinatorics ensure that if $|V| = r$, we may take $|W| \leq$

$\binom{2r-2}{r-1}$, as we formulate the finite Ramsey problem as a two-player game, it is not the sizes $|V|$ and $|W|$ Alice and Bob will be playing, but rather Alice will play a coloring g, and Bob will play a particular homogeneous set H.

Definition 4.8.1. $\mathsf{FinRam}(2,2)$ is the problem (A, B, R) where $A = B = \mathbb{N}^{<\mathbb{N}}$, and $\bar{a}\ R\ \bar{b}$ *if*, when $\bar{a}$ has length r and the elements of $\bar{a}$ encode a coloring $g : K(W_r) \to 2$ of the complete graph on $W_r = \left\{0, \ldots, \binom{2r-2}{r-1} - 1\right\}$, *then* $\bar{b}$ encodes a subset $H \subseteq W$, with $|H| = r$ that is homogeneous with respect to g; i.e., $\operatorname{ran} g \restriction K(H) = \{c\}$ for some color $c < 2$.

The encoding of g is a length-r vector where coordinate $i < r$ encodes a coloring of $K(W_{i+1})\backslash K(W_i)$. This encoding ensures that the problem is bounded as well as solvable.

$\mathsf{SeqFinRam}(2,2)$ is defined in the usual way. We will see that $\mathsf{SeqFinRam}(2,2)$ is equivalent to $\mathsf{ACA_0}$. The reader should not confuse this principle with RT^2_2, the infinite Ramsey's Theorem for Pairs, which is famously incomparable with $\mathsf{WKL_0}$ as described in Section 1.1. We are considering the sequential version of the finite Ramsey's theorem. In fact, this is one of the easiest applications to analyze, since the closed kernel is never solvable.

Theorem 4.8.2. $\mathsf{FinRam}(2,2)$ *does not have a solvable closed kernel.*

Proof. Define $\bar{a}$ by:

$$\begin{aligned} a_0 &= \varnothing \\ a_1 &= \{((0,1), \text{blue})\} \\ a_2 &= \{((i,j), \text{red}) : 0 \le i < j \le 5,\ (i,j) \ne (0,1)\} \end{aligned}$$

Note that $a_0 = K(\{0\})$, $a_0 \cup a_1 = K(\{0,1\})$, and $a_0 \cup a_1 \cup a_2 = K(\{0,1,2,3,4,5\})$, and the sizes of the respective sets W_1, W_2, W_3 are $\binom{2(1)-2}{1-1} = 1$, $\binom{2(2)-2}{2-1} = 2$, $\binom{2(3)-2}{3-1} = 6$.

Suppose that $\bar{b}$ is a vector of length 3 that gives a homogeneous set. Then the possible choices for $\bar{b}$ are $\langle 2,3,4\rangle$, $\langle 2,3,5\rangle$, $\langle 2,4,5\rangle$, $\langle 3,4,5\rangle$. However, for $\bar{a} \upharpoonright 2$, the only possible response from Bob is $\bar{b} = \langle 0,1\rangle$, which is not an initial segment of any of his four possible choices.

□

Corollary 4.8.3 ($\mathsf{RCA_0}$)**.** $\mathsf{SeqFinRam}(2,2) \leftrightarrow \mathsf{ACA_0}$.

Proof. This follows from Proposition 3.3.3 and Theorem 4.8.2.

□

Since the finite Ramsey's Theorem for $k > 2$ and/or for n-tuples $n > 2$ easily implies the finite Ramsey's Theorem for pairs on 2 colors, we can conclude that $\mathsf{SeqFinRam}(n,k) \leftrightarrow \mathsf{ACA_0}$ for any $n, k \geq 2$. The precise formulation of the problem must be modified, particularly the bounds. The proof, however, is near-identical, just with different bounds involved, and with $[W_r]^n$ replacing $K(W_r)$.

4.9 Separating Sets

$\mathsf{WKL_0}$ is well known to be equivalent to Σ^0_1-separation, which states: "Given two injections f, g with disjoint ranges, there exists a separating set X such that ran $f \subseteq X$ and
ran $g \subseteq (\mathbb{N} - X)$." In this section we consider the problem of finding a separating set for two finite functions, and the related sequential problem.

Proposition 4.9.1. *The following are equivalent over* $\mathsf{RCA_0}$*:*

(i) $\mathsf{ACA_0}$

(ii) Let $\langle f_n, g_n \rangle_{n\in\mathbb{N}}$ be a sequence of pairs of functions, both of whose domains are finite, and whose ranges are disjoint. Then there is a sequence $\langle X_n \rangle_{n\in\mathbb{N}}$ of finite separating sets such that $\forall n\ (ran\, f_n \subseteq X_n\ \wedge\ ran\, g_n \subseteq \mathbb{N} - X_n)$.

Proof. (i) $\Rightarrow$ (ii) by Proposition 2.1.4. Assume (ii), and let A be an oracle. We will show that the halting set relative to A exists, implying $\mathsf{ACA_0}$.

Define $\langle f_n \rangle_{n\in\mathbb{N}}$ as follows: $f(0) = 0$. If $a > 0$, then if there is s in the interval $\lfloor \frac{a}{2} \rfloor \leq s \leq a - 1$ such that $\Phi^A_n(n) \downarrow [s]$ for the first time, then $a \in \operatorname{dom} f_n$ and $f_n(a) = a - s + 1$. We define $\langle g_n \rangle_{n\in\mathbb{N}}$ as follows: $g_n(0) = 1$, and if a is such that if $s = \lfloor \frac{a}{2} \rfloor - 1$, and $\Phi^A_n(n) \downarrow [s]$ for the first time, then define $g_n(a) = a - s + 1$.

For instance, if $\Phi^A_n(n)$ halts for the first time at stage 5, then we have $f_n(0) = 0$, $f_n(6) = 2$, $f_n(7) = 3$, $f_n(8) = 4$, $f_n(9) = 5$, $f_n(10) = 6$, $f_n(11) = 7$, and we have $g_n(0) = 1$, $g_n(12) = 8$. It is easy to see that if $\Phi^A_n(n)$ halts for the first time at stage s, then the range of f_n will be $\{0, 2, 3, \ldots, s+2\}$, and the range of g_n will be $\{1,\ s+3\}$.

By (ii), there is a sequence $\langle X_n \rangle_{n\in\mathbb{N}}$ of finite sets separating the ranges of f_n, g_n. To test whether $\Phi^A_n(n)$ halts, find the smallest integer $j > 1$ not in X_n. (We are given that X_n is finite, so this is possible.) If $\Phi^A_n(n)$ does not halt at some stage s, $0 \leq s < j$, then $\Phi^A_n(n)$ does not halt. Thus we can compute the Turing jump A', implying $\mathsf{ACA_0}$.

□

In the above proof, there is no uniform bound on the domains of the functions f_n, nor is there a uniform bound on the *size* of their domains. If we bound the size of the domains of f_n and g_n, it gets a bit more interesting.

Unlike in the rest of this chapter, we will formulate these sequential problems directly rather than using the language of Chapter 3. The semi-bounded condition is problematic, as it is hard to put any kind of bound on the separating sets.

Definition 4.9.2. SBSS (Sequential Bounded Separating Sets) is the following statement: Let $\langle f_n, g_n \rangle_{n\in\mathbb{N}}$ be a sequence of pairs of functions such that

$$\exists b\, \forall n\, (|\operatorname{dom} f_n| \leq b \ \wedge \ |\operatorname{dom} g_n| \leq b \ \wedge \ (\operatorname{ran} f_n \cap \operatorname{ran} g_n = \varnothing)).$$

Then there is a sequence $\langle X_n \rangle_{n\in\mathbb{N}}$ of finite separating sets such that

$$\forall n\, (\operatorname{ran} f_n \subseteq X_n \ \wedge \ \operatorname{ran} g_n \subseteq \mathbb{N} - X_n).$$

Definition 4.9.3. SBCS (Sequential Bounded Containing Sets) is the following statement: Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of functions such that $\exists b\, \forall n\, (|\operatorname{dom} f_n| \leq b)$. Then there is a sequence $\langle X_n \rangle_{n\in\mathbb{N}}$ of finite containing sets such that $\forall n\, (\operatorname{ran} f_n \subseteq X_n)$.

Definition 4.9.4. SBCS' is SBCS together with the conclusion that $|X_n| \leq b$ for all n.

Clearly both SBSS and SBCS' imply SBCS.

In all definitions above it is the *cardinalities* of the domains that are bounded. If we were simply bounding the domains, all three statements would easily have constructive proofs in $\mathsf{RCA_0}$.

Proposition 4.9.5 ($\mathsf{RCA_0}$)**.** *$\mathsf{SBSS} \vdash \mathsf{WKL_0}$.*

Proof. Assume SBSS. We will prove Σ^0_1 separation, which implies $\mathsf{WKL_0}$. Let f, g be two injections with disjoint ranges. Define $\langle f_n, g_n \rangle_{n\in\mathbb{N}}$ as follows: $f_n(0) = 0$, and if there exists $k > 0$ with $f(k) = n$, then $f_n(k+1) = 2$. $g_n(0) = 1$, and if there exists $k > 0$ with $g(k) = n$, then $g_n(k+1) = 2$.

Note that both f, g have domains of size either 1 or 2, and since f, g have disjoint ranges, we will never have both $2 \in \operatorname{ran} f_n$ and $2 \in \operatorname{ran} g_n$. So the hypotheses of

SBSS are satisfied. Let $\langle X_n \rangle_{n\in\mathbb{N}}$ be the guaranteed sequence of finite separating sets. Define a function $h : \mathbb{N} \to \mathbb{N}$ by $h(n) = 1$ if $2 \in X_n$, and $h(n) = 0$ if $2 \notin X_n$. Then if $n \in \operatorname{ran} f$, $h(n) = 1$, and if $n \in \operatorname{ran} g$, $h(n) = 0$. So h separates the ranges of f and g.

□

Proposition 4.9.6 (RCA_0). *The following are equivalent:*

(i) WKL_0

(ii) If $f : \mathbb{N} \to 2$ is partial computable, then f has a total extension.

(iii) Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of functions such that $\forall n\, (|\operatorname{dom} f_n| \leq 1)$. Then there is a sequence $\langle X_n \rangle_{n\in\mathbb{N}}$ of sets with $\forall n\, (|X_n| \leq 1 \ \wedge\ \operatorname{ran} f_n \subseteq X_n)$.

Proof. (i) $\Leftrightarrow$ (ii) is well known and very straightforward to prove.

(ii) $\Rightarrow$ (iii): Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be given as in (iii). Define a partial computable function $g \subseteq: \mathbb{N}^2 \to \mathbb{N}$ as follows:

$$g(n,m) = \begin{cases} 1 & \text{if } \exists k\, (f_n(k) = m) \\ 0 & \text{if } \exists k\, \exists m'\, (m \neq m' \ \wedge f_n(k) = m') \end{cases}$$

Let $h : \mathbb{N}^2 \to \mathbb{N}$ be a total extension of g. Define

$$X_n = \{m \, : \, h(n,m) = 1 \wedge (\forall k < m)\, h(n,k) = 0\,\}$$

Then $\langle X_n \rangle_{n\in\mathbb{N}}$ is our desired sequence for statement (iii).

(iii) $\Rightarrow$ (ii): Let $f : \mathbb{N} \to 2$ be partial computable. Define $f_n := f|_{\{n\}}$, so that $|\operatorname{dom} f_n| \leq 1$. Let $\langle X_n \rangle$ be the sequence of containing sets guaranteed by (iii). Then define g by $g(n) = 1$ if $1 \in X_n$; $g(m) = 0$ if $0 \in X_n$; $g(n) = 1$ otherwise; then g is a total extension of f.

□

Proposition 4.9.7 ($\mathsf{RCA_0}$)**.** $\mathsf{SBCS}' \leftrightarrow \mathsf{WKL_0}$.

Proof. Statement (iii) in Proposition 4.9.6 is just SBCS' for $b = 1$. Therefore, SBCS' implies (iii) which implies $\mathsf{WKL_0}$.

Now assume $\mathsf{WKL_0}$, and we will prove SBCS'. Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of functions whose domains have size at most b. For each $n \in \mathbb{N}$, we inductively define $h_n^{(i)}$, $1 \le i \le b$, and sets $X_n^{(i)}$, $1 \le i \le b$, as follows: We define

$$\begin{aligned} h_n^{(i)}(k) = m \quad \Leftrightarrow \quad & f_n(k) = m \wedge (\forall i' < i)\, m \notin X_n^{(i')} \\ & \wedge (\forall (k', m') < (k, m)) \left[(\forall i' < i)(m' \notin X_n^{(i')}) \rightarrow f_n(k') \neq m' \right] \end{aligned}$$

It is clear from our definition that $\forall i\, (|\operatorname{dom} h_n^{(i)}| \le 1)$. Therefore, by (i) $\Rightarrow$ (iii) in Proposition 4.9.6, there exists a sequence of sets $\langle X_n^{(i)} \rangle_{n\in\mathbb{N}}$ of sets with $\forall n\, (|X_n^{(i)}| \le 1 \wedge \operatorname{ran} h_n^{(i)} \subseteq X_n^{(i)})$. We then use $X_n^{(i)}$ to define $h_n^{(i+1)}$ and so on. Let $X_n = X_n^{(1)} \cup \cdots \cup X_n^{(b)}$. Then we have $|X_n| \le b$ and $\operatorname{ran} f_n \subseteq X_n$, as desired.

□

Proposition 4.9.8 ($\mathsf{RCA_0}$)**.** $\mathsf{WKL_0} \vdash \mathsf{SBSS}$.

Proof. Assume $\mathsf{WKL_0}$. Then by Proposition 4.9.7, SBCS' also holds. Let $\langle f_n, g_n \rangle_{n\in\mathbb{N}}$ be sequences of functions with $|\operatorname{dom} f_n| \le b$, $|\operatorname{dom} g_n| \le b$, and $\operatorname{ran} f_n \cap \operatorname{ran} g_n = \varnothing$ for all n, as in the hypothesis of SBSS. Let $\langle X_n \rangle_{n\in\mathbb{N}}$ be the finite containing sets for $\langle f_n \rangle_{n\in\mathbb{N}}$ guaranteed by SBCS', so that $\operatorname{ran} f_n \subseteq X_n$.

Define a partial function $h \subseteq: \mathbb{N}^2 \to 2$ by: $h(n, m) = 0$ if $m \notin X_n$; $h(n, m) = 0$ if $m \in X_n$ and $\exists k (g_n(k) = m)$; $h(n, m) = 1$ if $m \in X_n$ and $\exists k (f_n(k) = m)$. Let $j : \mathbb{N}^2 \to 2$ be a total extension of h; this is guaranteed by the (i) $\Rightarrow$ (ii) implication in Proposition 4.9.6. Define a sequence of separating sets $\langle Y_n \rangle_{n\in\mathbb{N}}$ by $m \in Y_n \Leftrightarrow$

$j(n, m) = 1$. Note that $\operatorname{ran} f_n \subseteq Y_n \subseteq X_n \backslash (\operatorname{ran} g_n)$, meaning that in particular Y_n is finite since X_n is finite. So $\langle Y_n \rangle_{n\in\mathbb{N}}$ is our desired sequence of separating sets. $\square$

Corollary 4.9.9 ($\mathsf{RCA_0}$). $\mathsf{WKL_0} \leftrightarrow \mathsf{SBSS} \leftrightarrow \mathsf{SBCS'} \rightarrow \mathsf{SBCS}$.

Therefore, $\mathsf{WKL_0} \vdash \mathsf{SBCS}$, but there is still the possibility that the implication is strict. If so, SBCS would be the only sequential problem considered in this thesis whose strength is not equivalent to one of $\mathsf{RCA_0}, \mathsf{WKL_0}$, or $\mathsf{ACA_0}$. This does not contradict all of the classification work from Chapter 3, since this formulation of the problem is not semi-bounded (unlike $\mathsf{SBCS'}$). The statement is certainly not provable in $\mathsf{RCA_0}$, as it is stronger than DNR:

Proposition 4.9.10 ($\mathsf{RCA_0}$). $\mathsf{SBCS} \vdash \mathsf{DNR}$.

Proof. Assume SBCS. Let an oracle A be given. Define $\langle f_n \rangle_{n\in\mathbb{N}}$ as follows: If $\Phi_n^A(n)\downarrow$ $[s]$ for the first time, then $f_n(s) = \Phi_n^A(n)$. If $\Phi_n^A(n)$ never halts, then f_n is the empty function. Let $\langle X_n \rangle_{n\in\mathbb{N}}$ be the sequence of finite containing sets guaranteed by SBCS. Define a function $g : \mathbb{N} \rightarrow \mathbb{N}$ by $g(n) :=$ the least positive integer not in X_n. Then we have $\forall n\, (g(n) \neq \Phi_n^A(n))$, as desired.

$\square$

Question 4.9.11. *What is the precise strength of* SBCS*?*

4.10 Summary of Applications

Let us recall, from Subsection 2.1.1, that if a sequential theorem requires $\mathsf{WKL_0}$, then any proof of the nonsequential version requires the Law of Excluded Middle (LEM). This points to a necessary non-uniformity in any proof of the nonsequential statement. Recall that $\mathsf{EL_0}$ is an intuitionistic analogue of $\mathsf{RCA_0}$ such that $\mathsf{RCA_0} = \mathsf{EL_0} + \mathsf{LEM}$.

Theorem 4.10.1 (Restatement of Theorem 2.1.8) (Dorais)**.** *Suppose that $\alpha(X)$ and $\beta(X,Z)$ are formulas that satisfy Condition Set Γ.*

(a) If

$$\mathsf{EL_0} \vdash \forall X(\alpha(X) \to \exists Z\beta(X,Z))$$

then

$$\mathsf{RCA_0} \vdash \forall X(\forall n\alpha(X_n) \to \exists Z\forall n\beta(X_n,Z_n))$$

.

(b) If

$$\mathsf{EL_0} + \mathsf{WKL} \vdash \forall X(\alpha(X) \to \exists Z\beta(X,Z))$$

then

$$\mathsf{WKL_0} + \mathsf{B\Sigma^0_2} \vdash \forall X(\forall n\alpha(X_n) \to \exists Z\forall n\beta(X_n,Z_n))$$

.

We restate Condition Set Γ as well:

Condition Set Γ**.** The following are conditions on $\alpha(X)$ and $\beta(X,Z)$ for Theorem 4.10.1 to hold. For part (a), $\alpha(X)$ must belong to the syntactic class N_K and $\beta(X,Z)$ must belong to the syntactic class Γ_K. For part (b), $\alpha(X)$ must belong to the syntactic class N_L and $\beta(X,Z)$ must belong to the syntactic class Γ_L. Definitions of these four syntactic classes can be found in Dorais [6].

As long as $\alpha(X)$ and $\beta(X,Z)$ assert that two finite sequences of rational numbers $\bar{a}$ and $\bar{b}$ are such that some computable relation holds ($\bar{a}\ R\ \bar{b}$), then Condition Set Γ holds.

Problem	Conditions	Description	Reference
$\mathsf{Col}(\mathcal{C}, r)$	$\mathcal{C}$ on-line r-colorable	Graph coloring	4.2.2
$\mathsf{Match}_k(\alpha, \beta)$	$\beta \leq \alpha/(1+\alpha)$ $k < \omega$ vertices	α-Hall marriage problem	4.3.2
$\mathsf{Sch}(N, k, \alpha)$	$\alpha \geq 1.923$	Task scheduling problem	4.4.3
$\mathsf{Page}(N, k, \alpha)$	$\alpha \geq k$	Paging problem	4.5.7
$\mathsf{LU}(N, k, \alpha)$	$\alpha \geq k$	List update problem	4.6.1

Table 4.1: Problems whose sequential versions are provable in RCA_0

Claim 4.10.2 (stated without proof)**.** *Let* $\mathsf{P} = (C, D, R)$ *be any problem that we have considered in Chapter 4. Then* P *can be expressed in the form* $\forall X(\alpha(X) \rightarrow \exists Z\beta(X, Z))$ *in such a way that* $\alpha(X)$ *and* $\beta(X, Z)$ *satisfy all parts of Condition* Γ*.*

The following three theorems — one for each of RCA_0, WKL_0, and ACA_0 — summarize our Chapter 4 results, and part (b) of Theorem 4.10.4 and part (d) of Theorem 4.10.5 express the results as relevant consequences of Theorem 4.10.1.

Theorem 4.10.3. *Let* P *be any problem in Table 4.1, and suppose that the corresponding conditions hold. Then* $\mathsf{RCA}_0 \vdash \mathsf{SeqP}$*.*

Theorem 4.10.4. *Let* P *be any problem in Table 4.2, and suppose that the corresponding conditions hold. Then*

(a) $\mathsf{RCA}_0 \vdash (\mathsf{SeqP} \leftrightarrow \mathsf{WKL}_0)$*.*

(b) $\mathsf{EL}_0 \nvdash \mathsf{P}$*. In particular,* LEM *is necessary for proving* P*.*

Theorem 4.10.5. *Let* P *be any problem in Table 4.3, and suppose that the corresponding conditions hold. Then*

(a) $\mathsf{RCA}_0 \vdash (\mathsf{SeqP} \leftrightarrow \mathsf{ACA}_0)$ *if* P *is not marked by a* $(\star)$ *in the table.*

(b) $(\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2) \vdash (\mathsf{SeqP} \leftrightarrow \mathsf{ACA}_0)$ *if* P *is marked by a* $(\star)$ *in the table.*

Problem	Conditions	Description	Reference
$\mathsf{Col}(\mathcal{C}, r)$	$\mathcal{C}$ not on-line r-colorable	Graph coloring	4.2.2
$\mathsf{Match}_k(\alpha, \beta)$	$\alpha/(1+\alpha) < \beta \leq \min(\alpha, 1)$ $k < \omega$ vertices	α-Hall marriage problem	4.3.2
$\mathsf{Page}(dk, k, \alpha)$	$1 \leq \alpha < k$, $dk < \omega$	Paging problem	4.5.7
$\mathsf{LU}(N, k, \alpha)$	$1 \leq \alpha < k$, $N, k < \omega$	List update problem	4.6.1
$\mathsf{Dilworth}(k)$	$k \geq 2$	Dilworth's theorem	4.7.1
$\mathsf{DualDilworth}(k)$	$k \geq 2$	Dual to Dilworth's theorem	4.7.6
SBSS		Bounded separating sets problem	4.9.2
SBCS'		Bounded containing sets problem, modified	4.9.4

Table 4.2: Problems whose sequential versions are equivalent to $\mathsf{WKL_0}$

(c) If $k < \omega$, then $\mathsf{RCA_0} \vdash (\mathsf{SeqP}_k \leftrightarrow \mathsf{ACA_0})$.

(c) $\mathsf{EL_0} + \mathsf{WKL} \nvdash \mathsf{P}$. *In particular,* LEM *is necessary for proving* P*, even assuming* WKL.

Problem	Conditions	Description	Reference
$\mathsf{PP}(k,p)$	$k \geq 2$, $p \in (0, \frac{1}{k}]$	Pigeonhole principle	4.1.1
$\star\mathsf{ThinPP}(k,p)$	$k \geq 2$, $p \in (0, 1-\frac{1}{k}]$	Thin pigeonhole principle	4.1.2
$\mathsf{Match}(\alpha,\beta)$	$\alpha/(1+\alpha) < \beta \leq \min(\alpha, 1)$ no bound on vertices	α-Hall marriage problem	4.3.2
$\star\mathsf{Sch}(N,k,\alpha)$	$\alpha \leq 1.852$[1], $40 \mid k$, $N = 4k+1$ or $\alpha = 1.88$, $k = 4$	Task scheduling problem	4.4.3
$\mathsf{FinRam}(n,k)$	$n, k \geq 2$	Finite Ramsey's theorem	4.8.1
		Unbounded separating sets problem	4.9.1

Table 4.3: Problems whose sequential versions are equivalent to ACA_0 over RCA_0. A star ($\star$) next to the problem name indicates that for nonstandard k, the problem is equivalent to ACA_0 over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$.

[1] If Conjecture 4.4.12 holds, then we can replace 1.852 with *any* α where $\mathsf{Sch}(N,k,\alpha)$ is not on-line solvable, for appropriate N, k.

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